QUASI-NORM SPACES(1)

BY

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Introduction. Let S be a linear space. A function q on S is called a *quasi-norm* if it satisfies:

(1) $0 \leqq q(x) \leqq +\infty$ for all $x \in S$,

(2) $q(0) = 0$,

(3) $|a| \leqq |b|$ implies that $q(ax) \leqq q(bx)$ for a, b real and $x \in S$,

(4) $q(x+y) \leqq q(x) + q(y)$ for all x and y in S.

A quasi-norm q is *proper* if $\lim_{a \to 0} q(ax) = 0$ for all $x \in S$. We say that a quasi-norm q_1 *dominates* another quasi-norm q_2 ($q_1 \succ q_2$) if, for every $\varepsilon > 0$ there is a $\delta > 0$, such that $q_1(x) < \delta$ implies $q_2(x) < \varepsilon$. A system $\boldsymbol{I}$ of quasi-norms is called an *ideal* if:

(1) $\boldsymbol{I} \ni q_1 \succ q_2$ implies $\boldsymbol{I} \ni q_2$,

(2) for any sequence $q_n \in \boldsymbol{I}$ $(n = 1, 2, \cdots)$ there is a quasi-norm $q \in \boldsymbol{I}$ such that $q \succ q_n$ for all $n = 1, 2, \cdots$.

A system $\boldsymbol{B}$ of quasi-norms is a basis for an ideal $\boldsymbol{I}$ if, for each $q_1 \in \boldsymbol{I}$ there is $q_2 \in \boldsymbol{B} \subset \boldsymbol{I}$ such that $q_1 \prec q_2$.

Given any linear space we show that there is a one-to-one correspondence between topologies on the linear space compatible with the linear operations and ideals of proper quasi-norms. Thus, study of ideals of proper quasi-norms will give us knowledge of linear topologies.

Given a set A such that $dA \subset A$ for $|d| \leqq 1$ we say that A is *of finite character c* if $bA + (1-b)A \subset cA$ for all b such that $0 < b < 1$. A set B is *bounded by a quasi-norm q* ($B \succ q$) if $\lim_{a \to 0} [\sup_{x \in B} q(ax)] = 0$. We show that if a set A is of finite character, then a quasi-norm q_2 can be constructed such that $A \succ q_2$ and $q_1 \prec q_2$ for any quasi-norm q_1 such that $A \succ q_1$. The quasi-norm constructed from A also has the following property: there are two numbers c, $d > 0$ such that $q(x) < d$ implies that $q((1/2c)x) \leqq \frac{1}{2} q(x)$. Quasi-norms that have this property are said to be *of finite character c*.

We investigate completeness and quasi-completeness of a linear topological space in terms of quasi-norms. We also show that a manifold is bounded by every quasi-norm in an ideal composed of proper quasi-norms if and only if it is bounded by the corresponding linear topology.

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A system F of quasi-norms is a *filter* if $F \ni q_1 \prec q_2$ implies $F \ni q_2$. We consider relations between ideals and filters. Given a subspace R of the linear space S we define the relative ideal on R of a given ideal on S. Given a directed system of subspaces and ideals (filters) on these subspaces, we define an inductive limit ideal (filter) on S. We examine what properties of the individual ideals the inductive limit inherits and look at the relation between inductive limits of filters and ideals. Next, we show that, on a finite-dimensional space, every ideal composed of proper quasi-norms is contained in an ideal which has a one-element basis.

In the latter part of the paper, quasi-norms on linear lattices are investigated. We consider various properties of quasi-norms which connect them with the lattice structure. A main result shows that a universally continuous linear lattice is super-universally continuous if there is a quasi-norm q defined on S which has the following properties:

(1) $|x| \leqq |y|$ implies $q(x) \leqq q(y)$ for all $x, y \in S$ (i.e. q is *monotone*).

(2) order-$\lim_{\nu \to \infty} a_\nu = 0$ implies $\lim_{\nu \to \infty} q(a_\nu) = 0$.

(3) $q(x) = 0$ implies $x = 0$.

The next object of study is ideals of quasi-norms having special properties. We prove that the following two properties are equivalent:

(1) I is a proper ideal which has a basis of monotone quasi-norms;

(2) the linear topology corresponding to I has a basis $\mathfrak{V}$ of neighborhoods of 0 which satisfy $|x| \leqq |y|$ and $y \in V \in \mathfrak{V}$ implies $x \in V$. The final important result says that if S is a universally continuous space and I is an ideal on S which has a basis of quasi-norms q such that $a_\lambda \uparrow_{\lambda \in \Lambda} a$ implies $q(a) = \sup_{\lambda \in \Lambda} q(a_\lambda)$ then $\{x : |x| \leqq |a|\}$ is complete for all $a \in S$.

1. **Quasi-norms.** Let S be a linear space. A function q defined on S is called a *quasi-norm* if:

(1) $0 \leqq q(x) \leqq +\infty$ for all $x \in S$,

(2) $|a| \leqq |b|$ implies $q(ax) \leqq q(bx)$,

(3) $q(x + y) \leqq q(x) + q(y)$,

(4) $q(0) = 0$.

Note that it is possible for a quasi-norm to assume infinite values. If $q(x) < +\infty$ for all $x \in S$ we say that q is a finite quasi-norm.

EXAMPLE 1. Define q^* on S by

$$q^*(x) = \begin{cases} 0 & \text{if } x = 0, \\ +\infty & \text{if } x \neq 0. \end{cases}$$

Then q^* is easily seen to be a quasi-norm.

EXAMPLE 2. If f is a linear functional on the linear space S then $|f|$ defined by $|f|(x) = |f(x)|$ is a finite quasi-norm.

Several properties of quasi-norms follow easily from the definition. Condition (2) shows that $q(ax) = q(|a|x)$ for all real a and $x \in S$.

By induction from condition (3) we conclude:

THEOREM 1.1. *$q(nx) \leqq nq(x)$ for all $n = 1,2,\cdots$.*

A quasi-norm q is said to be *proper* if $\lim_{a\to 0} q(ax) = 0$ for all $x \in S$. The quasi-norm defined in Example 1 is not proper while the one in Example 2 is proper.

THEOREM 1.2. *If q is a proper quasi-norm, then q is finite.*

Proof. Since, for arbitrary $x \in S$, $\lim_{a\to 0} q(ax) = 0$ there is $\delta > 0$ such that $q(ax) \leqq 1$ for $|a| \leqq \delta$. Choose a positive integer n such that $1 < n\delta$. Then $q(x) \leqq q((n\delta)x) \leqq nq(\delta x) \leqq n < +\infty$.

THEOREM 1.3. *For any quasi-norm q, if we set $q^\alpha(x) = \min\{q(x),\alpha\}$ for $\alpha > 0$, we obtain a finite quasi-norm q^α. If q is proper, then q^α is also proper.*

Proof. Clearly $0 \leqq q^\alpha(x) \leqq \alpha$ for all $x \in S$. If $|a| \leqq |b|$ then

$$\begin{aligned} q^\alpha(ax) &= \min\{q(ax),\alpha\} \leqq \min\{q(bx),\alpha\} = q^\alpha(bx), \\ q^\alpha(x+y) &= \min\{q(x+y)\alpha\} \leqq \min\{q(x)+q(y),\alpha\} \\ &\leqq \min\{q(x)+q(y), q(x)+\alpha,\ q(y)+\alpha,\ \alpha+\alpha\} \\ &= \min\{q(x),\alpha\} + \min\{q(y),\alpha\} = q^\alpha(x)+q^\alpha(y), \\ q^\alpha(0) &= \min\{q(0),\alpha\} = 0. \end{aligned}$$

If q is proper, then we have:

$$0 \leqq \lim_{a\to 0} q^\alpha(ax) \leqq \lim_{a\to 0} q(ax) = 0.$$

Theorem 1.2 shows that every proper quasi-norm is finite but the converse is false. If q^* is defined as in Example 1, then $(q^*)^1$ is a finite quasi-norm which is not proper.

THEOREM 1.4. *For any system q_λ $(\lambda \in \Lambda)$ of quasi-norms and for any system a_λ $(\lambda \in \Lambda)$ of positive real numbers the function $\sum_{\lambda\in\Lambda} a_\lambda q_\lambda$ given by: $(\sum_{\lambda\in\Lambda} a_\lambda q_\lambda)(x) = \sup_{\text{finite} H\subset\Lambda} \sum_{\lambda\in H} a_\lambda q_\lambda(x)$ is also a quasi-norm. Furthermore, if q_λ is proper for all $\lambda \in \Lambda$ and $\sum_{\lambda\in\Lambda} a_\lambda q_\lambda(x)$ is finite for all $x \in S$ then $\sum_{\lambda\in\Lambda} a_\lambda q_\lambda$ is a proper quasi-norm.*

Proof. Clearly $0 \leqq \sum_{\lambda\in\Lambda} a_\lambda q_\lambda(x) \leqq +\infty$. $|a| \leqq |b|$ implies $\sum_{\lambda\in H} a_\lambda q_\lambda(ax) \leqq \sum_{\lambda\in H} a_\lambda q_\lambda(bx)$ for every finite $H \subset \Lambda$. Thus

$$\sum_{\lambda\in\Lambda} a_\lambda q_\lambda(ax) \leqq \sum_{\lambda\in\Lambda} a_\lambda q_\lambda(bx),$$

$$\begin{aligned}\sum_{\lambda\in\Lambda} a_\lambda q_\lambda(x+y) &= \sup_{\text{finite} H\subset\Lambda} \sum_{\lambda\in H} a_\lambda q_\lambda(x+y)\\ &\leqq \sup_{\text{finite } H\subset\Lambda}\left\{\sum_{\lambda\in H} a_\lambda q_\lambda(x)+\sum_{\lambda\in H} a_\lambda q_\lambda(y)\right\}\\ &\leqq \sup_{\text{finite } H\subset\Lambda}\sum_{\lambda\in H} a_\lambda q_\lambda(x)+\sup_{\text{finite } H\subset\Lambda}\sum_{\lambda\in H} a_\lambda q_\lambda(y)\\ &= \sum_{\lambda\in\Lambda} a_\lambda q_\lambda(x)+\sum_{\lambda\in\Lambda} a_\lambda q_\lambda(y).\end{aligned}$$

Clearly $\sum_{\lambda\in\Lambda} a_\lambda q_\lambda(0)=0$.

Now suppose that q_λ is proper for all $\lambda\in\Lambda$ and that $\sum_{\lambda\in\Lambda} a_\lambda q_\lambda$ is finite. Given any $x\in S$ and any $\varepsilon>0$ there is a finite set $H_0\subset\Lambda$ such that

$$\sum_{\lambda\in H_0} a_\lambda q_\lambda(x) > \sum_{\lambda\in\Lambda} a_\lambda q_\lambda(x)-\varepsilon.$$

But if we note that:

$$\begin{aligned}\sum_{\lambda\in\Lambda} a_\lambda q_\lambda(x) &= \sup_{\text{finite } H\subset\Lambda}\sum_{\lambda\in H} a_\lambda q_\lambda(x)\\ &= \sup_{\text{finite } H\subset\Lambda}\sum_{\lambda\in H\cup H_0} a_\lambda q_\lambda(x)\\ &= \sum_{\lambda\in H_0} a_\lambda q_\lambda(x)+\sum_{\lambda\in\Lambda-H_0} a_\lambda q_\lambda(x),\end{aligned}$$

we see $\sum_{\lambda\in H_0} a_\lambda q_\lambda(x) > \sum_{\lambda\in H_0} a_\lambda q_\lambda(x)+\sum_{\lambda\in\Lambda-H_0} a_\lambda q_\lambda(x)-\varepsilon$ and this implies $\sum_{\lambda\in\Lambda-H_0} a_\lambda q_\lambda(x)<\varepsilon$. Therefore we have:

$$\begin{aligned}\lim_{a\to 0}\left(\sum_{\lambda\in\Lambda} a_\lambda q_\lambda(ax)\right) &= \lim_{a\to 0}\left(\sum_{\lambda\in H_0} a_\lambda q_\lambda(ax)+\sum_{\lambda\in\Lambda-H_0} a_\lambda q_\lambda(ax)\right)\\ &\leqq \lim_{a\to 0}\sum_{\lambda\in H_0} a_\lambda q_\lambda(ax)+\varepsilon = \varepsilon.\end{aligned}$$

Since ε was arbitrary $\lim_{a\to 0}\sum_{\lambda\in\Lambda} a_\lambda q_\lambda(ax)=0$.

For two quasi-norms q_1 and q_2 we write $q_1 \prec q_2$ if, for any $\varepsilon>0$ there is $\delta>0$ such that $q_2(x)<\delta$ implies $q_1(x)<\varepsilon$.

The next three theorems follow easily from this definition.

THEOREM 1.5. *If q_1, q_2 and q_3 are quasi-norms such that $q_1\prec q_2$ and $q_2\prec q_3$, then $q_1\prec q_3$.*

THEOREM 1.6. *If q_1 and q_2 are quasi-norms such that $q_1\prec q_2$, then $aq_1\prec bq_2$ for any $a,b>0$.*

THEOREM 1.7. *If q_1 and q_2 are quasi-norms such that $q_1(x)\leqq q_2(x)$ for all $x\in S$ then $q_1\prec q_2$.*

If we denote by 0 the quasi-norm defined by $0(x) = 0$ for all $x \in S$ and let q^* be defined as in Example 1 then, for every quasi-norm q, $0(x) \leqq q(x) \leqq q^*(x)$ for all $x \in S$. Thus, $0 \prec q \prec q^*$ for every quasi-norm q.

THEOREM 1.8. *If q_1 and q_2 are quasi-norms such that $q_1 \prec q_2$ and q_2 is proper, then q_1 is also proper.*

Proof. Given $x \in S$ and $\varepsilon > 0$ there is $\delta_1 > 0$ such that $q_2(y) < \delta_1$ implies $q_1(y) < \varepsilon$. Since q_2 is proper there is $\delta_2 > 0$ such that $|a| < \delta_2$ implies $q_2(ax) < \delta_1$. Thus $|a| < \delta_2$ implies $q_1(ax) < \varepsilon$. Therefore q_1 is proper.

THEOREM 1.9. *If q_1, q_2 and q_3 are quasi-norms such that $q_1 \succ q_2$ and $q_1 \succ q_3$, then $q_1 \succ q_2 + q_3$.*

Proof. For any $\varepsilon > 0$ we can find δ_1 and δ_2, both positive, such that $q_1(x) < \delta_1$ implies $q_2(x) < \varepsilon/2$ and $q_1(x) < \delta_2$ implies $q_3(x) < \varepsilon/2$. Then for $q_1(x) < \min\{\delta_1,\delta_2)$ we have $q_2(x) + q_3(x) < \varepsilon$.

By induction we can extend this theorem to any finite sum but a slight modification is necessary in order to extend the theorem to an infinite sum.

THEOREM 1.10. *If $q \prec q_\lambda$ for all $\lambda \in \Lambda$ and if $\Sigma_{\lambda \in \Lambda}(\sup_{x \in S} q_\lambda(x)) < +\infty$ then $q \succ \Sigma_{\lambda \in \Lambda} q_\lambda$.*

Proof. Given $\varepsilon > 0$ since $\Sigma_{\lambda \in \Lambda}(\sup_{x \in S} q_\lambda(x)) < +\infty$ there is a finite $H_0 \subset \Lambda$ such that:

$$\sum_{\lambda \in H_0} \left(\sup_{x \in S} q_\lambda(x)\right) > \sum_{\lambda \in \Lambda} \sup_{x \in S} q_\lambda(x)) - \varepsilon/2$$

$$= \sum_{\lambda \in H_0} \sup_{x \in S} q_\lambda(x)] + \sum_{\lambda \in \Lambda - H_0} \sup_{x \in S} q_\lambda(x) - \varepsilon/2.$$

Therefore, $\Sigma_{\lambda \in \Lambda - H_0} q_\lambda(x) < \Sigma_{\lambda \in \Lambda - H_0} \sup_{x \in S} q_\lambda(x) < \varepsilon/2$ for all $x \in S$. Since H_0 is finite there is $\delta > 0$ such that $q(x) < \delta$ implies $\Sigma_{\lambda \in H_0} q_\lambda(x) < \varepsilon/2$. Thus, if $q(x) < \delta$ we have:

$$\sum_{\lambda \in \Lambda} q_\lambda(x) = \sum_{\lambda \in H_0} q_\lambda(x) + \sum_{\lambda \in \Lambda - H_0} q_\lambda(x) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Therefore $q \succ \Sigma_{\lambda \in \Lambda} q_\lambda$.

2. **Bounded manifolds.** If A is a manifold of the linear space S and if q is a quasi-norm on S we define $q(A) = \sup_{x \in A} q(x)$. The following relations follow easily.

(1) $q(aA) \leqq q(bA)$ for $|a| \leqq |b|$,

(2) $q(-A) = q(A)$,

(3) $q(A + B) \leqq q(A) + q(B)$.

A manifold A is said to be *bounded by a quasi-norm* q (written $A] \succ q$)if

$\lim_{a\to 0} q(aA) = 0$. With this definition we see easily that if $A \succ q$, then $aA \succ q$ for any real a. Also it is clear that $A \succ q_1$ and $q_1 \succ q_2$ implies that $A \succ q_2$.

THEOREM 2.1. *If A is a manifold of S and q is a quasi-norm, then $A \succ q$ implies that $\bigcup_{|a|\leqq 1} aA \succ q$.*

Proof. $q(\bigcup_{|a|\leqq 1} aA) \leqq \sup_{|a|\leqq 1} q(aA) \leqq q(A)$. Therefore, for any real b:

$$q\left(b\left\{\bigcup_{|a|\leqq 1} aA\right\}\right) \leqq q(bA)$$

and the last term goes to zero as b does.

THEOREM 2.2. *If A and B are manifolds which are bounded by a quasi-norm q, then $A + B$ and $A \cup B$ are also bounded by q.*

Proof. $\lim_{a\to 0} q(a(A + B)) \leqq \lim_{a\to 0} \{q(aA) + q(aB)\} = 0$.

$$\lim_{a\to 0} q(a(A \cup B)) = \lim_{a\to 0} q(aA \cup aB)$$
$$= \lim_{a\to 0}\max \{q(aA), q(aB)\} = \max\left\{\lim_{a\to 0} q(aA), \lim_{a\to 0} q(aB)\right\} = 0.$$

A manifold A is said to be *symmetric* if $A = -A$. A manifold A is said to be *star* if $A \supset aA$ for $0 \leqq a \leqq 1$.

It is easily seen that, for any manifold A, the manifold $\bigcup_{|a|\leqq 1} aA$ is the smallest symmetric star manifold containing A.

A manifold A is said to be a *character manifold* (or A is said to be *of finite character*) if it is symmetric and star and satisfies the following: there is a positive real number c such that $aA + bA \subset cA$ for $a + b \leqq 1$; $a, b \geqq 0$. Such a c is called a *character of A* and A is said to be *of finite character c*.

THEOREM 2.3. *If A_λ $(\lambda \in \Lambda)$ is a system of manifolds such that each one is of character c, then $A_0 = \bigcap_{\lambda\in\Lambda} A_\lambda$ is also of character c.*

Proof. Let x, $y \in A_0$. Then, for $a + b \leqq 1$; $a, b \geqq 0$ we have $ax + by \in cA_\lambda$ for all $\lambda \in \Lambda$ which implies $ax + by \in \bigcap_{\lambda\in\Lambda} cA_\lambda = cA_0$. Therefore, $aA_0 + bA_0 \subset cA_0$.

Since a character manifold is star by definition, we see that $0 < a < b$ implies that $aA \subset bA$ for any character manifold A. Thus, if c is a character of A, then any larger number is also a character of A.

THEOREM 2.4. *A manifold A has a character less than one if and only if A is a linear manifold.*

Proof. If A is a linear manifold $aA + bA \subset A = \frac{1}{2}A$ for a, $b \geqq 0$, $a + b \leqq 1$. If A has character $c < 1$, we first show that, for $a > 0$, $aA = A$. Let $x \in A$. Then $x = \frac{1}{2}x + \frac{1}{2}x = cz_1$ for $z_1 \in A$. By induction we define a sequence $z_\nu \in A\,(\nu = 1, 2, \cdots)$ such that $z_\nu = \frac{1}{2}z_\nu + \frac{1}{2}z_\nu = cz_{\nu+1}$. Thus $x = c^\nu z_\nu$ (for $\nu = 1, 2, \cdots$). Choose ν_0

such that $c^{\nu_0}a < 1$. Then $ax = (ac^{\nu_0})z_{\nu_0} \in A$ since A is star. Thus $aA = A$ for $a > 0$. If $a < 0$, $aA = (-a)(-A) = (-a)A = A$ since A is symmetric. Given $x, y \in A$ we have $x + y = 2(\frac{1}{2}x + \frac{1}{2}y) \in 2(cA) = (2c)A = A$.

This result shows that the linear space S itself is a character manifold and that any positive number is a character of S.

Given a manifold A and a positive real number c, the character c hull of A is defined as the intersection of all manifolds of character c containing A. (This system is nonempty since S itself is of character c.) We denote the character c hull of A by $\bar{A}^c$.

Clearly $\bar{A}^c = \overline{(\bigcup_{|a| \leqq 1} aA)}^c$ since a manifold of character c is star and symmetric. If A is of character c, then dA is of character c for every real d since, for $a + b \leqq 1$; $a, b \geqq 0$, $a(dA) + b(dA) = d(aA + bA) \subset d(cA) = c(dA)$. For any manifold A consider $\overline{aA}^c$ where a is any real number $\neq 0$. B is a manifold of character c containing A if and only if aB is a manifold of character c containing aA. Therefore $(\overline{aA})^c = \bigcap_{aA \subset D \text{ of char.} c} D = \bigcap_{aA \subset aB \text{ of char.} c} aB = a\bar{A}^c$.

A quasi-norm q is said to be *of finite character* if there are positive real numbers a and c such that $q(x) \leqq a$ implies that $\frac{1}{2}q(x) \geqq q((1/2c)x)$. If q satisfies this requirement, we say that q is *of character* c. Clearly if q is of character c, then q is of character c' for any $c' > c$.

THEOREM 2.5. *If q_1 and q_2 are quasi-norms of finite character, then $aq_1 + bq_2$ is also of finite character for a, $b > 0$.*

Proof. Let d_i, c_i $(i = 1, 2)$ be such that $q_i(x) \leqq d_i$ implies $\frac{1}{2}q_i(x) \geqq q_i((1/2c_i)x)$. $(i = 1, 2)$. Setting $d_0 = \min\{d_1, d_2\}$ and $c_0 = \max\{c_1, c_2\}$ we find that $aq_1(x) + bq_2(x) \leqq \min\{ad_0, bd_0\}$ implies that $q_1(x) \leqq d_0$ and $q_2(x) \leqq d_0$. Therefore, $\frac{1}{2}aq_1(x) \geqq aq_1((1/2c_1)x)$ and $\frac{1}{2}bq_2(x) \geqq bq_2((1/2c_2)x)$. Hence,

$$\tfrac{1}{2}(aq_1(x) + bq_2(x)) \geqq aq_1((1/2c_0)x) + bq_2((1/2c_0)x).$$

THEOREM 2.6. *A quasi-norm q is of finite character if and only if q^α is of finite character for all $\alpha > 0$.*

Proof. If q is of finite character, then there are a_0, c_0 such that $q(x) \leqq a_0$ implies $\frac{1}{2}q(x) \geqq q((1/2c_0)x)$. Putting $a_1 = \min\{a_0, \alpha/2\}$ and $c_1 = c_0$, we have that $q^\alpha(x) \leqq a_1$ implies $q(x) = q^\alpha(x) \leqq a_1 \leqq a_0$. Therefore, $\frac{1}{2}q^\alpha(x) = \frac{1}{2}q(x) \geqq q((1/2c_1)x) = q^\alpha((1/2c_1)x)$ since $q((1/2c_1)x) \leqq \frac{1}{2}q(x) = \frac{1}{2}q^\alpha(x) \leqq \frac{1}{2}\alpha$.

If q^α is of finite character for some $\alpha > 0$, then there are positive real numbers a and c such that $q^\alpha(x) \leqq a$ implies $\frac{1}{2}q^\alpha(x) \geqq q^\alpha((1/2c)x)$. If we let $a_0 = \min\{a, \alpha/2\}$, we have that $q(x) \leqq a_0$ implies $q(x) = q\ (x) \leqq a$ which gives $\frac{1}{2}q(x) = \frac{1}{2}q^\alpha(x) \geqq q^\alpha((1/2c)x) = q((1/2c)x)$.

THEOREM 2.7. *If $A \succ q$ and q is a quasi-norm of finite character c then $\bar{A}^{2c} \succ q$.*

Proof. Since q is of character c, we can find $a_0 > 0$ such that $q(x) \leqq a_0$ implies

$\frac{1}{2}q(x) \geqq q((1/2c)x)$. Hence, if $q(x) \leqq a_0$ and $q(y) \leqq a_0$, then for $a+b \leqq 1$; $a, b \geqq 0$ we have:

$$q((1/2c)(ax+by)) \leqq q((a/2c)x) + q((b/2c)y)$$
$$\leqq q((1/2c)x) + q((1/2c)y) \leqq \tfrac{1}{2}q(x) + \tfrac{1}{2}q(y) \leqq a_0.$$

Therefore, if we let $V_{a_0} = \{x : q(x) \leqq a_0\}$, we have: $aV_{a_0} + bV_{a_0} \subset (2c)V_{a_0}$ for $a+b \leqq 1$; $a, b \geqq 0$. Thus, $2c$ is a character of V_{a_0}. Now since $A \succ q$, there is $d > 0$ such that $q(dA) \leqq a_0$ or, equivalently $dA \subset V_{a_0}$. The fact that V_{a_0} is of character $2c$ shows us that $d\bar{A}^{2c} = \overline{dA^{2c}} \subset V_{a_0}$. Thus, we need only show that V_{a_0} is bounded by q in order to have $d\bar{A}^{2c}$ (and therefore $\bar{A}^{2c} = ((1/d)d\bar{A}^{2c})$ also bounded by q.

Given $\varepsilon > 0$ choose ν_0 such that $a_0/2^{\nu_0} < \varepsilon$ and let $x \in V_{a_0}$. Then

$$q((1/(2c)^{\nu_0})x) \leqq \tfrac{1}{2}q((1/(2c)^{\nu_0-1})x) \leqq \cdots \leqq (1/2^{\nu_0})q(x) < \varepsilon.$$

Hence $q((1/(2c)^{\nu_0})V_{a_0}) < \varepsilon$ and $V_{a_0} \succ q$.

A manifold V of the linear space S is called a *vicinity* if, for $x \in S$ we can find $b > 0$ such that $ax \in V$ for $0 \leqq a \leqq b$.

The following theorem is proved in [3].

THEOREM 2.8 ([3], p. 129). *If a symmetric star vicinity V is of finite character c (i.e. $aV + bV \subset cV$ for $a+b \leqq 1$; $a, b \geqq 0$) then there is a proper quasi-norm q on S such that*

$$\{x : q(x) < 1/2^\nu\} \subset (1/2c)^\nu V \subset \{x : q(x) \leqq 1/2^\nu\}$$

and

$$(q(1/2c)x) = \tfrac{1}{2}q(x) \text{ for } q(x) \leqq \tfrac{1}{2}.$$

(Note that the definition of quasi-norm in [3] corresponds to a proper quasi-norm in our terminology.)

In the proof of the preceding theorem the fact that the set V is a vicinity is used only to establish that the quasi-norm is proper. Since every character manifold is star and symmetric, the rest of the proof gives the following result:

THEOREM 2.9. *Let A be a manifold of finite character c. Then there exists a not necessarily proper quasi-norm q of finite character c such that:*

$$\left\{x : q(x) < \frac{1}{2^\nu}\right\} \subset \left(\frac{1}{2c}\right)^\nu A \subset \left\{x : q(x) \leqq \frac{1^\cdot}{2^\nu}\right\}$$

and

$$q((1/2c)x) = \tfrac{1}{2}q(x) \textit{ for } q(x) \leqq \tfrac{1}{2}.$$

This quasi-norm is called the quasi-norm associated with A and denoted by q_A.

THEOREM 2.10. *If A is a manifold of finite character, then $A \succ q_A$. Furthermore, if q is any quasi-norm such that $A \succ q$ then we have $q \prec q_A$.*

Proof. If A is of character c, then $(1/2c)^{\nu}A \subset \{x: q_A(x) \leqq 1/2^{\nu}\}$ which shows that $A \succ q_A$.

If q is a quasi-norm such that $A \succ q$ then, for any $\varepsilon > 0$ there is a $\delta > 0$ such that $q(\delta A) < \varepsilon$. Choose ν_0 such that $(1/2c)^{\nu_0} < \delta$. Then $q_A(x) < 1/2^{\nu_0}$ implies that $x \in (1/2c)^{\nu_0} A \subset \delta A$ which implies $q(x) < \varepsilon$. Therefore $q \prec q_A$.

3. **Ideals.** A system $\boldsymbol{I}$ of quasi-norms is called an *ideal* if:

(1) $\boldsymbol{I} \ni q_1 \succ q_2$ implies $\boldsymbol{I} \ni q_2$.

(2) for any sequence $q_\nu \in \boldsymbol{I}$ $(\nu = 1, 2, \cdots)$ there is $q \in \boldsymbol{I}$ such that $q_\nu \prec q$ for all $\nu = 1, 2, \cdots$.

Since $0 \prec q$ for all quasi-norms q we see that every ideal $\boldsymbol{I} \ni 0$.

THEOREM 3.1. *$q_\nu \in \boldsymbol{I}$ for all $\nu = 1, 2, \cdots$ implies that $\sum_{\nu=1}^{\infty} q_\nu^{1/2^\nu} \in \boldsymbol{I}$.*

Proof. By property (2) there is $q \in \boldsymbol{I}$ such that $q_\nu \prec q$ for all $\nu = 1, 2, \cdots$. But since $\sum_{\nu=1}^{\infty} \sup_{x \in S} q_\nu^{1/2^\nu}(x) \leqq \sum_{\nu=1}^{\infty} (\frac{1}{2})^\nu < +\infty$, Theorem 1.10 shows that $\sum_{\nu=1}^{\infty} q_\nu^{1/2^\nu} \prec q$. Hence $\sum_{\nu=1}^{\infty} q_\nu^{1/2^\nu} \in \boldsymbol{I}$ by (1) above.

A subsystem $\boldsymbol{B} \subset \boldsymbol{I}$ is called a *basis* if, for any $q_1 \in \boldsymbol{I}$ there is $q_2 \in \boldsymbol{B}$ such that $q_2 \succ q_1$. Clearly every basis satisfies the *basis condition*: for any sequence $q_\nu \in \boldsymbol{B}$ $(\nu = 1, 2, \cdots)$ there is $q \in \boldsymbol{B}$ such that $q_\nu \prec q$ for all $\nu = 1, 2, \cdots$. Conversely we have:

THEOREM 3.2. *If a system $\boldsymbol{B}$ of quasi-norms satisfies the basis condition, then there exists uniquely an ideal $\boldsymbol{I}$ such that $\boldsymbol{B}$ is a basis of $\boldsymbol{I}$.*

Proof. Let $\boldsymbol{I} = \{q: q \prec q_1$ for some $q_1 \in \boldsymbol{B}\}$. Then if $\boldsymbol{I} \ni q_1 \succ q_2$ it is clear that $\boldsymbol{I} \ni q_2$. If $\boldsymbol{I} \ni q_\nu$ $(\nu = 1, 2, \cdots)$ then $q_\nu \prec q'_\nu \in \boldsymbol{B}$ $(\nu = 1, 2, \cdots)$. But then there is $q \in \boldsymbol{B}$ such that $q'_\nu \prec q$ $(\nu = 1, 2, \cdots)$. $q \in \boldsymbol{B}$ implies $q \in \boldsymbol{I}$ and $q \succ q'_\nu \succ q_\nu$ shows that $q \succ q_\nu$ $(\nu = 1, 2, \cdots)$. Thus, $\boldsymbol{I}$ is an ideal. If any other ideal $\boldsymbol{I}_1$ contains $\boldsymbol{B}$ it must contain $\boldsymbol{I}$. But if $\boldsymbol{B}$ is a basis for $\boldsymbol{I}_1$, then $q_1 \in \boldsymbol{I}_1$ implies $q_1 \prec q_2$ for some $q_2 \in \boldsymbol{B}$ and hence $q_1 \in \boldsymbol{I}$. Therefore, $\boldsymbol{I}_1 = \boldsymbol{I}$ and $\boldsymbol{I}$ is unique.

THEOREM 3.3. *If $\boldsymbol{B}$ is a basis for an ideal $\boldsymbol{I}$, then, for any $\alpha > 0$, $\{q^\alpha: q \in \boldsymbol{B}\}$ is also a basis for $\boldsymbol{I}$.*

Proof. $q \geqq q^\alpha$ implies $q \succ q^\alpha$ and this shows that $q^\alpha \in \boldsymbol{I}$ for all $q \in \boldsymbol{B}$. We need only show that $q \prec q^\alpha$ to conclude $\{q^\alpha: q \in \boldsymbol{B}\}$ is a basis. For any $\varepsilon > 0$ if we choose $\delta = \min\{\varepsilon, \alpha/2\}$ we have $q^\alpha(x) < \delta$ implies $q(x) = q^\alpha(x) < \delta \leqq \varepsilon$.

We say that an ideal $\boldsymbol{I}_1$ is *stronger* than another ideal $\boldsymbol{I}_2$ or $\boldsymbol{I}_2$ is *weaker* than $\boldsymbol{I}_1$ if $\boldsymbol{I}_1 \supset \boldsymbol{I}_2$.

THEOREM 3.4 *If $\boldsymbol{I}_\lambda$ $(\lambda \in \Lambda)$ is a system of ideals, then $\boldsymbol{I}_0 = \bigcap_{\lambda \in \Lambda} \boldsymbol{I}_\lambda$ is an ideal which is the strongest ideal among all those weaker than all $\boldsymbol{I}_\lambda$ $(\lambda \in \Lambda)$.*

Proof. We need only show $\boldsymbol{I}_0$ is an ideal and the rest is clear. If $\boldsymbol{I}_0 \ni q_1 \succ q_2$, then we see that $\boldsymbol{I}_\lambda \ni q_1 \succ q_2$ for all $\lambda \in \Lambda$ which means that $q_2 \in \boldsymbol{I}_\lambda$ for all $\lambda \in \Lambda$.

Therefore $q_2 \in \boldsymbol{I}_0$. If $q_\nu \in \boldsymbol{I}_0$ $(\nu = 1,2,\cdots)$, then $q_\nu \in \boldsymbol{I}_\lambda$ for all $\lambda \in \Lambda$, $\nu = 1,2,\cdots$. Hence, $\sum_{\nu=1}^\infty q_\nu^{1/2^\nu} \in \boldsymbol{I}_\lambda$ for all $\lambda \in \Lambda$ which implies $\sum_{\nu=1}^\infty q_\nu^{1/2^\nu} \in \boldsymbol{I}_0$ and clearly $\sum_{\nu=1}^\infty q_\nu^{1/2^\nu} \succ q_\nu^{1/2^\nu} \succ q_\nu$ $(\nu = 1,2,\cdots)$.

THEOREM 3.5. *If $\boldsymbol{I}_\lambda$ $(\lambda \in \Lambda)$ is a system of ideals, then there is a weakest ideal among all those stronger than all $\boldsymbol{I}_\lambda$ $(\lambda \in \Lambda)$. We denote this ideal by $\bigvee_{\lambda \in \Lambda} \boldsymbol{I}_\lambda$. It has a basis given by*

$$\boldsymbol{B} = \left\{ \sum_{\nu=1}^\infty q_\nu : \sum_{\nu=1}^\infty \sup_{x \in S} q_\nu(x) < +\infty,\ q_\nu \in \bigcup_{\lambda \in \Lambda} \boldsymbol{I}_\lambda,\ \nu = 1,2\cdots \right\}.$$

Another basis is given by

$$\boldsymbol{B}_1 = \left\{ \sum_{\nu=1}^\infty q_\nu^{1/2^\nu} : q_\nu \in \bigcup_{\lambda \in \Lambda} \boldsymbol{I}_\lambda \text{ for } \nu = 1,2,\cdots \right\}.$$

Proof. If $\boldsymbol{I}$ is an ideal containing $\bigcup_{\lambda \in \Lambda} \boldsymbol{I}_\lambda$ and if $\sum_{\nu=1}^\infty q_\nu \in \boldsymbol{B}$, then there is $q \in \boldsymbol{I}$ such that $q \succ q_\nu$ $(\nu = 1,2,\cdots)$. But then we have, by Theorem 1.10, $q \succ \sum_{\nu=1}^\infty q_\nu$ which implies $\sum_{\nu=1}^\infty q_\nu \in \boldsymbol{I}$. Thus, $\boldsymbol{I} \supset \boldsymbol{B}$. If we can show that $\boldsymbol{B}$ satisfies the basis condition, it will be clear that the ideal generated by $\boldsymbol{B}$ is the weakest stronger ideal. Suppose $p_\nu \in \boldsymbol{B}$ $(\nu = 1,2,\cdots)$. Then $p_\nu = \sum_{\mu=1}^\infty q_{\nu,\mu}$ where $\sum_{\mu=1}^\infty \sup_{x \in S} q_{\nu,\mu}(x) \leqq \alpha_\nu < +\infty$ for $\nu = 1, 2, \cdots$. Now consider $\sum_{\nu,\mu} (1/\alpha_\nu 2^\nu) q_{\nu,\mu}$. Then

$$\sum_{\nu,\mu} \sup_{x \in S} \frac{1}{\alpha_\nu 2^\nu} q_{\nu,\mu}(x) = \sum_{\nu=1}^\infty \frac{1}{\alpha_\nu 2^\nu} \sum_{\mu=1}^\infty \sup_{x \in S} q_{\nu,\mu}(x) \leqq 1.$$

Therefore $\sum_{\nu,\mu} (1/\alpha_\nu 2^\nu) q_{\nu,\mu} \in \boldsymbol{B}$ and clearly $\sum_{\nu,\mu} (1/\alpha_\nu 2^\nu) q_{\nu,\mu} \succ p_\nu$ $(\nu = 1,2,\cdots)$ since

$$\sum_{\nu,\mu} \frac{1}{\alpha_\nu 2^\nu} q_{\nu,\mu} \succ q_{\nu,\mu} \ (\mu = 1,2,\cdots)\ (\nu = 1,2,\cdots).$$

Now, as for $\boldsymbol{B}_1$, if we can show that for any $q \in \boldsymbol{B}$ there is $q_1 \in \boldsymbol{B}$, such that $q \prec q_1$, we will have shown that $\boldsymbol{B}_1$ is also a basis since it is clear that $\boldsymbol{B}_1 \subset \boldsymbol{B}$. Suppose $q = \sum_{\nu=1}^\infty q_\nu \in \boldsymbol{B}$. Let $q_1 = \sum_{\nu=1}^\infty q_\nu^{1/2^\nu} \in \boldsymbol{B}_1$. Since $\sum_{\nu=1}^\infty q_\nu^{1/2^\nu} \succ q_\nu^{1/2^\nu} \succ q_\nu$ for all $\nu = 1,2,\cdots$ we have that $q_1 = \sum_{\nu=1}^\infty q_\nu^{1/2^\nu} \succ \sum_{\nu=1}^\infty q_\nu = q$ by Theorem 1.10.

An ideal $\boldsymbol{I}$ is said to be *proper* if every $q \in \boldsymbol{I}$ is proper. Since $q \prec q_1$ proper implies q is proper, we need only know that a basis of $\boldsymbol{I}$ consists entirely of proper quasi-norms to conclude $\boldsymbol{I}$ is proper.

4. **Induced linear topologies.** Linear topologies were defined first by Kolmogoroff [2]. After that, von Neumann gave another definition in [7]. In [3] it is proved that these two definitions are equivalent. We will use the definition of von Neumann.

A system $\mathfrak{V}$ of vicinities in the linear space S is called a *linear topology* on S if:

(1) $V \in \mathfrak{V}$, $V \subset U$ implies $U \in \mathfrak{V}$,

(2) U, $V \in \mathfrak{V}$ implies $U \cap V \in \mathfrak{V}$,

(3) $V \in \mathfrak{V}$ implies $aV \in \mathfrak{V}$ for all $a \neq 0$,

(4) for any $V \in \mathfrak{V}$ we can find $U \in \mathfrak{V}$ such that $aU \subset V$ for $0 \leqq a \leqq 1$,

(5) for any $V \in \mathfrak{V}$ we can find $U \in \mathfrak{V}$ such that $U + U \subset V$.

As on p. 139 of [3] we can consider, for every vicinity V on S, the corresponding connector (connector is defined on p. 62 of [3]) V_c defined by $V_c(x) = V + x$. Then, if $\mathfrak{V}$ is a linear topology, it follows easily that $\{V_c : V \in \mathfrak{V}\}$ is a basis for a uniformity on S which is called the *induced uniformity* by $\mathfrak{V}$ and denoted by $\mathfrak{U}^{\mathfrak{V}}$. $\mathfrak{U}^{\mathfrak{V}}$, in turn, induces a topology on S which is called the induced topology by $\mathfrak{V}$ and denoted $\mathfrak{T}^{\mathfrak{V}}$. If $\mathfrak{T}$ is a topology on S induced by a linear topology $\mathfrak{V}$ we refer to S with the topology $\mathfrak{T}$ as the *linear topological space* $(S,\mathfrak{T})$.

Theorem 4.1. *A quasi-norm q on S is uniformly continuous by the induced uniformity $\mathfrak{U}^{\mathfrak{V}}$ if and only if, for any $\varepsilon > 0$ there is $V \in \mathfrak{V}$ such that $q(V) < \varepsilon$. If q is uniformly continuous it is proper.*

Proof. We first prove the "if" part. Given $\varepsilon > 0$ by hypothesis we can choose $V \in \mathfrak{V}$ such that $q(V) < \varepsilon$. Then if $x \in V_c(y) = V + y$, we have $x - y \in V$. Then $|q(x) - q(y)| \leqq q(x - y) < \varepsilon$ and we see that q is uniformly continuous. The fact that $|q(x) - q(y)| \leqq q(x - y)$ shows that a quasi-norm is uniformly continuous by $\mathfrak{U}^{\mathfrak{V}}$ if it is just continuous at zero by $\mathfrak{T}^{\mathfrak{V}}$.

If q is uniformly continuous then, for any $\varepsilon > 0$, there is $V \in \mathfrak{V}$ such that $x \in V_c(0) = 0 + V$ implies $|q(x) - q(0)| = q(x) < \varepsilon$.

To show that a uniformly continuous quasi-norm is proper, suppose we are given $\varepsilon > 0$. Then there is $V \in \mathfrak{V}$ such that $q(V) < \varepsilon$. Since V is a vicinity, given any $x \in S$ there is $b > 0$ such that $ax \in V$ for $0 \leqq a \leqq b$. Therefore $q(ax) < \varepsilon$ for $0 \leqq a \leqq b$ and hence $\lim_{a \to 0} q(ax) = 0$. Since x was arbitrary q is proper.

Theorem 4.2. *If $(S,\mathfrak{T})$ is a linear topological space, the system of all quasi-norms on S which are continuous by $\mathfrak{T}$ is a proper ideal which we denote $\boldsymbol{I}(S,\mathfrak{T})$. Furthermore, if $\boldsymbol{I}(S,\mathfrak{T}_1) = \boldsymbol{I}(S,\mathfrak{T}_2)$ then $\mathfrak{T}_1 = \mathfrak{T}_2$.*

Proof. By Theorem 4.1 the system is composed of proper quasi-norms, and we must show that it is an ideal. If q_1 is continuous and $q_2 \prec q_1$ then, for any $\varepsilon > 0$ there is $\delta > 0$ such that $q_1(x) < \delta$ implies $q_2(x) < \varepsilon$. But for such a δ we can find $V \in \mathfrak{V}$ such that $q_1(V) < \delta$. Then, $q_2(V) < \varepsilon$ and hence q_2 is continuous by Theorem 4.1. If $q_1, q_2, \cdots$ is a sequence of continuous quasi-norms, let $p = \sum_{\nu=1}^{\infty} q_\nu^{1/2^\nu}$. Then $p \succ q_\nu^{1/2^\nu} \succ q$ for all $\nu = 1, 2, \cdots$ and we need only show that p is continuous. Given $\varepsilon > 0$ choose ν_0 so that $\sum_{\nu=\nu_0+1}^{\infty} 1/2^\nu < \varepsilon/2$. For each q_ν, $\nu = 1, 2, \cdots, \nu_0$ we can find $V_\nu \in \mathfrak{V}$ such that $q_\nu(V_\nu) < (\varepsilon/2\nu_0)$. Let $V_0 = \bigcap_{\nu=1}^{\nu_0} V_\nu \in \mathfrak{V}$. Then $p(V_0) = \sup_{x \in V_0} p(x) \leqq \sum_{\nu=1}^{\nu_0} q_\nu(V_0) + \varepsilon/2 \leqq \sum_{\nu=1}^{\nu_0} (\varepsilon/2\nu_0) + \varepsilon/2 = \varepsilon$. Thus p is continuous.

Now suppose $I(S,\mathfrak{T}_1)=I(S,\mathfrak{T}_2)$ and let $\mathfrak{T}_i$ be induced by $\mathfrak{V}_i$ $(i=1,2)$. Given $V\in\mathfrak{V}_1$ we can choose a sequence $V_\nu\in\mathfrak{V}_1$ $(\nu=1,2,\cdots)$ of star symmetric neighborhoods of zero such that $V_1\subset V$ and $V_{\nu+1}+V_{\nu+1}\subset V_\nu$ $(\nu=1,2,\cdots)$. Then using the construction given in Theorem 3 on p. 129 of [3], we can construct a quasi-norm q such that

$$\left\{x:q(x)<\frac{1}{2^\nu}\right\}\subset V_\nu\subset\left\{x:q(x)\leqq\frac{1}{2^\nu}\right\}.$$

Since each $V_\nu\in\mathfrak{V}_1$ we see that $q\in I(S,\mathfrak{T}_1)$. By assumption then, $q\in I(S,\mathfrak{T}_2)$ which implies that $\{x:q(x)<\frac{1}{2}\}\in\mathfrak{V}_2$. Since this set is contained in V we see that $V\in\mathfrak{V}_2$. Therefore $\mathfrak{T}_1\subset\mathfrak{T}_2$ and symmetry shows that $\mathfrak{T}_1=\mathfrak{T}_2$.

THEOREM 4.3. *If I is any proper ideal on S, then $I=I(S,\mathfrak{T})$ for some topology $\mathfrak{T}$.*

Proof. For any $q\in I$ let $V_\alpha=\{x:q(x)\leqq\alpha\}$ for $\alpha>0$. In §63 of [3] it is shown that $\mathfrak{V}_q=\{V:V\supset V_\alpha \text{ for some } \alpha>0\}$ is a linear topology. Let $\mathfrak{V}^q$ be the topology induced by $\mathfrak{V}_q$ and consider $\mathfrak{T}_0=\bigvee_{q\in I}\mathfrak{T}^q$. It is not difficult to show that $(S,\mathfrak{T}_0)$ is a linear topological space.

If $q\in I$ then, by the definition of $\mathfrak{T}^q$, q is continuous by $\mathfrak{T}^q$. Since $\mathfrak{T}^q\subset\mathfrak{T}_0$ we see that q is continuous by $\mathfrak{T}_0$. Therefore, $I\subset I(S,\mathfrak{T}_0)$.

Now let $p\in I(S,\mathfrak{T}_0)$. Since p is continuous, for each $\nu=1,2,\cdots$ we can find $V_\nu\in\mathfrak{T}_0$ such that $p(V_\nu)<1/\nu$. By the construction of $\mathfrak{T}_0=\bigvee_{q\in I}\mathfrak{T}^q$ we see that for each V_ν there is a set $q_{\nu,\mu}\in I$ $(\mu=1,2,\cdots,m_\nu)$ and $\delta_\nu>0$ such that $V_\nu\supset\{x:q_{\nu,\mu}(x)<\delta_\nu;\ \mu=1,2,\cdots,m_\nu\}$ $(\nu=1,2,\cdots)$. Since I is an ideal $\sum_{\mu=1}^{m_\nu}q_{\nu,\mu}=p_\nu\in I$ and clearly $V_\nu\supset\{x:p_\nu(x)<\delta_\nu\}$. Now let $q=\sum_{\nu=1}^{\infty}p_\nu^{1/2^\nu}\in I$. Given $\varepsilon>0$ there is ν_0 such that $1/\nu_0<\varepsilon$. Let $\delta=\min\{1/2^{\nu_0},\delta_{\nu_0}\}$. Then $q(x)<\delta$ implies $p_{\nu_0}(x)<\delta_{\nu_0}$ and thus, $p(x)<1/\nu_0<\varepsilon$. Hence, $p\prec q$ which implies $p\in I$. Therefore $I(S,\mathfrak{T}_0)\subset I$.

Theorems 4.2 and 4.3 together show that there is a one-to-one correspondence between linear topologies and proper ideals. The next theorem shows that this correspondence is order-preserving.

THEOREM 4.4. *If $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are topologies on S such that $(S,\mathfrak{T}_1)$ and $(S,\mathfrak{T}_2)$ are linear topological spaces, then $\mathfrak{T}_1\subset\mathfrak{T}_2$ if and only if $I(S,\mathfrak{T}_1)\subset I(S,\mathfrak{T}_2)$.*

Proof. If $\mathfrak{T}_1\subset\mathfrak{T}_2$, then we see that q continuous by $\mathfrak{T}_1$ implies q continuous by $\mathfrak{T}_2$ and therefore $I(S,\mathfrak{T}_1)\subset I(S,\mathfrak{T}_2)$.

Conversely, if $I(S,\mathfrak{T}_1)\subset I(S,\mathfrak{T}_2)$, then $q\in I(S,\mathfrak{T}_1)\subset I(S,\mathfrak{T}_2)$ implies that $\mathfrak{T}^q\subset\bigvee_{q\in I(S,\mathfrak{T}_2)}\mathfrak{T}^q$ which means that

$$\bigvee_{q\in I(S,\mathfrak{T}_1)}\mathfrak{T}^q=\mathfrak{T}_1\subset\mathfrak{T}_2=\bigvee_{q\in I(S,\mathfrak{T}_2)}\mathfrak{T}^q.$$

Given a proper ideal $\boldsymbol{I}$ we denote the topology associated with $\boldsymbol{I}$ by $\mathfrak{T}^{\boldsymbol{I}}$. This topology is induced by a linear topology which we denote by $\mathfrak{V}_{\boldsymbol{I}}$.

5. **Completeness.** Let $\mathfrak{V}$ be a linear topology. A manifold A of S is said to be *bounded by* $\mathfrak{V}$ if, for any $V \in \mathfrak{V}$ there is $a > 0$ such that $aA \subset V$.

THEOREM 5.1. *If $\mathfrak{V}_{\boldsymbol{I}}$ is the linear topology associated with a proper ideal $\boldsymbol{I}$ and A is a manifold of S then A is bounded by $\mathfrak{V}_{\boldsymbol{I}}$ if and only if $A \succ q$ for all $q \in \boldsymbol{I}$.*

Proof. If A is bounded by $\mathfrak{V}_{\boldsymbol{I}}$ then for any $q \in \boldsymbol{I}$ and $\varepsilon > 0$ there is $V \in \mathfrak{V}_{\boldsymbol{I}}$ such that $q(V) < \varepsilon$. Then there is $a > 0$ such that $aA \subset V$ and hence $q(aA) < \varepsilon$.

If $A \succ q$ for all $q \in \boldsymbol{I}$, let $V \in \mathfrak{V}_{\boldsymbol{I}}$. Then there is $q \in \boldsymbol{I}$ such that $\{x : q(x) < 1\} \subset V$. But then there is $\delta > 0$ such that $q(\delta A) < 1$. Therefore $\delta A \subset V$.

THEOREM 5.2. *If A is bounded by $\mathfrak{V}$, then the closure of A in the topology $\mathfrak{T}^{\mathfrak{V}}$ is also bounded by $\mathfrak{V}$.*

Proof. For any $q \in \boldsymbol{I}(S, \mathfrak{T}^{\mathfrak{V}})$ and any $\varepsilon > 0$ there is $a > 0$ such that $q(aA) < \varepsilon$. Since q is continuous $q(aA^{-}) = q((aA)^{-}) \leqq \varepsilon$. Thus $A^{-} \succ q$ for all $q \in \boldsymbol{I}(S, \mathfrak{T}^{\mathfrak{V}})$ and consequently A^{-} is bounded by $\mathfrak{V}$ by Theorem 5.1.

THEOREM 5.3. *$S \ni x_\delta \to_{\delta \in \Delta} x$ by $\mathfrak{V}_{\boldsymbol{I}}$ (that is, for any $V \in \mathfrak{V}_{\boldsymbol{I}}$ there is δ_0 belonging to the directed system Δ such that $\delta \leqq \delta_0$ implies $x_\delta \in x + V$) if and only if $q(x_\delta - x) \to_{\delta \in \Delta} 0$ for all $q \in \boldsymbol{B}$ where $\boldsymbol{B}$ is a basis of $\boldsymbol{I}$.*

Proof. If $x_\delta \to_{\delta \in \Delta} x$ by $\mathfrak{V}_{\boldsymbol{I}}$, then given any $q \in \boldsymbol{B}$ and any $\varepsilon > 0$ there is $V \in \mathfrak{V}_{\boldsymbol{I}}$ such that $q(V) < \varepsilon$. But $x_\delta \to_{\delta \in \Delta} x$ means there is $\delta_0 \in \Delta$ such that $x_\delta - x \in A$ for all $\delta \leqq \delta_0$. Hence, $q(x_\delta - x) < \varepsilon$ for all $\delta \leqq \delta_0$.

As for the converse, if we are given any $V \in \mathfrak{V}_{\boldsymbol{I}}$, there is a quasi-norm $q_V \in \boldsymbol{I}$ such that $\{x : q_V(x) < 1\} \subset V$. Since $\boldsymbol{B}$ is a basis of $\boldsymbol{I}$, there is $q \in \boldsymbol{B}$ such that $q_V \prec q$. Hence, there is $a > 0$ such that $q(x) < a$ implies $q_V(x) < 1$. Since $q(x_\delta - x) \to_{\delta \in \Delta} 0$, there is $\delta_0 \in \Delta$ such that $q(x_\delta - x) < a$ for $\delta \leqq \delta_0$. Hence, $\delta \leqq \delta_0$ implies $q(x_\delta - x) < a$ which means $q_V(x_\delta - x) < 1$ and hence $x_\delta - x \in V$. Therefore $x_\delta \to_{\delta \in \Delta} x$ by $\mathfrak{V}_{\boldsymbol{I}}$.

A system $S \ni x_\delta\ (\delta \in \Delta)$ is a *Cauchy system by* $\mathfrak{V}$ if, for any $V \in \mathfrak{V}$ there is $\delta_0 \in \Delta$ such that $x_{\delta_1} - x_{\delta_2} \in V$ for $\delta_1, \delta_2 \leqq \delta_0$.

THEOREM 5.4. *A directed system $S \ni x_\delta\ (\delta \in \Delta)$ is a Cauchy system by $\mathfrak{V}_{\boldsymbol{I}}$ if and only if $q(x_{\delta_1} - x_{\delta_2}) \to_{\delta_1, \delta_2 \in \Delta} 0$ for all $q \in \boldsymbol{B}$ where $\boldsymbol{B}$ is a basis of $\boldsymbol{I}$.*

Proof. Follows the same pattern as the proof of Theorem 5.3.

We say that a linear space S with linear topology $\mathfrak{V}$ is *complete* if, for every Cauchy system $x_\delta\ (\delta \in \Delta)$ there is $x \in S$ such that $x_\delta \to_{\delta \in \Delta} x$.

THEOREM 5.5. *S is complete by $\mathfrak{V}_{\boldsymbol{I}}$ if and only if $q(x_{\delta_1} - x_{\delta_2}) \to_{\delta_1, \delta_2 \in \Delta} 0$ for all $q \in \boldsymbol{B}$ (a basis of $\boldsymbol{I}$) implies that $q(x_\delta - x) \to_{\delta \in \Delta} 0$ for some $x \in S$ and all $q \in \boldsymbol{B}$.*

Proof. This is an immediate consequence of the two preceding theorems.

On p. 155 of [3] it is shown that, with every linear space S with linear topology $\mathfrak{V}$, we can associate another linear space $\bar{S}$ with linear topology $\bar{\mathfrak{V}}$ having the following properties:

(1) $\bar{S}$ is complete by $\bar{\mathfrak{V}}$,

(2) S is a linear manifold of $\bar{S}$,

(3) $\mathfrak{V}$ is the relative linear topology of $\bar{\mathfrak{V}}$,

(4) S is dense in $\bar{S}$ by $\mathfrak{T}^{\bar{\mathfrak{V}}}$,

(5) $\{0\}^{\mathfrak{V}-} \subset S$.

$\bar{S}$ is called the *completion* of S and is unique up to isomorphism. Using Theorem 6 on p. 90 of [3], we see that every continuous quasi-norm q on S can be extended to a function $\bar{q}$ on $\bar{S}$ which is continuous by $\bar{\mathfrak{V}}$. Since the real numbers form a Hausdorff space $\bar{q}$ is uniquely determined. It can be easily shown that $\bar{q}$ is actually a quasi-norm on $\bar{S}$. Thus, if $\boldsymbol{I}$ is an ideal of quasi-norms on S, we can correspond the ideal $\bar{\boldsymbol{I}} = \{\bar{q} : q \in \boldsymbol{I}\}$ on $\bar{S}$. ($\bar{\boldsymbol{I}}$ can easily be shown to be an ideal if we note that $q_1 \prec q_2$ implies $\bar{q}_1 \prec \bar{q}_2$. This follows from the relation:

$$\{x \in \bar{S} : \bar{q}_2(x) < \delta\} \subset \{x \in S : q_2(x) < \delta\}^- \subset \{x \in S : q_1(x) < \varepsilon\}^- \subset \{x \in \bar{S} : \bar{q}_1(x) \leqq \varepsilon\}.)$$

If q is a quasi-norm on $\bar{S}$, we can consider the quasi-norm q^S on S defined by $q^S(x) = q(x)$ for $x \in S$. Then, if $\boldsymbol{I}'$ is an ideal on $\bar{S}$, we correspond the ideal $\boldsymbol{I} = \{q^S : q \in \boldsymbol{I}'\}$. Then we see that $\bar{\boldsymbol{I}} = \boldsymbol{I}'$ and we have a one-to-one correspondence between ideals on S and ideals on $\bar{S}$.

A linear space S with linear topology $\mathfrak{V}$ is said to be *conditionally complete* if the closure of every set which is bounded by $\mathfrak{V}$ is complete by $\mathfrak{U}^{\mathfrak{V}}$.

THEOREM 5.6. *A linear space S with linear topology $\mathfrak{V}_I$ is conditionally complete if and only if $q(x_{\delta_1} - x_{\delta_2}) \to_{\delta_1, \delta_2 \in \Delta} 0$ and $\{x_\delta : \delta \in \Delta\} \succ q$ for all $q \in \boldsymbol{B}$, a basis of $\boldsymbol{I}$ implies there is $x \in S$ such that $q(x_\delta - x) \to_{\delta \in \Delta} 0$ for all $q \in \boldsymbol{B}$.*

Proof. If S is conditionally complete the result is clear. Conversely, if A is bounded by $\mathfrak{V}_I$ and $A \ni x_\delta$ $(\delta \in \Delta)$ then $\{x_\delta : \delta \in \Delta\} \subset A$ implies $\{x_\delta : \delta \in \Delta\} \succ q$ for all $q \in \boldsymbol{B}$. Theorems 5.3 and 5.4 then show that there is $x \in A^-$ such that $x_\delta \to_{\delta \in \Delta} x$.

6. **Filters.** A system $\boldsymbol{F}$ of quasi-norms is called a *filter* if $\boldsymbol{F} \ni q \prec q_1$ implies $\boldsymbol{F} \ni q_1$.

Given any system $\boldsymbol{B}$ of quasi-norms if we set $\boldsymbol{F} = \{q : q \succ q_1$ for all $q_1 \in \boldsymbol{B}\}$ then $\boldsymbol{F}$ is a filter. We denote this filter by $\boldsymbol{B}^{\mathfrak{F}}$ and call it the *associated filter of* $\boldsymbol{B}$. If we set $\boldsymbol{I} = \{q : q \prec q_1$ for all $q_1 \in \boldsymbol{B}\}$ then we obtain an ideal $\boldsymbol{I}$. We denote this ideal $\boldsymbol{B}^{\mathfrak{J}}$ and call it the *associated ideal of* $\boldsymbol{B}$. (The fact that $\boldsymbol{B}^{\mathfrak{J}}$ *is* an ideal is a consequence of the fact that $q_\nu \prec q$ for all $\nu = 1, 2, \cdots$ implies that $\sum_{\nu=1}^{\infty} q_\nu^{1/2^\nu} \prec q$.)

A filter F (ideal I) is called *reflexive* if $F^{\mathfrak{J}\mathfrak{F}} = F$ ($I^{\mathfrak{F}\mathfrak{J}} = I$.)

THEOREM 6.1. *For any system B, $B^{\mathfrak{J}}$ and $B^{\mathfrak{F}}$ are reflexive.*

Proof. Note first that, for any system C, $C \subset C^{\mathfrak{J}\mathfrak{F}}$ *and* $C \subset C^{\mathfrak{F}\mathfrak{J}}$. Thus, $B^{\mathfrak{J}} \subset (B^{\mathfrak{J}})^{\mathfrak{F}\mathfrak{J}}$. But on the other hand, $B \subset B^{\mathfrak{J}\mathfrak{F}}$ implies that $B^{\mathfrak{J}} \supset B^{\mathfrak{J}\mathfrak{F}\mathfrak{J}}$. Thus $B^{\mathfrak{J}} = B^{\mathfrak{J}\mathfrak{F}\mathfrak{J}}$ and a similar argument shows that $B^{\mathfrak{J}} = B^{\mathfrak{F}\mathfrak{J}\mathfrak{F}}$.

A manifold A is said to be *bounded by an ideal I* (we write $A \succ I$) if $A \succ q$ for all $q \in I$.

THEOREM 6.2. *If A is a character manifold and $A \succ I$ then there is $q_1 \in I^{\mathfrak{F}}$ such that $A \succ q_1$.*

Proof. If we let $q_1 = q_A$, the quasi-norm defined in Theorem 2.9, then Theorem 2.10 shows that $q_1 \in I^{\mathfrak{F}}$ and $A \succ q_1$.

Note that the converse holds even when A is not a character manifold. In fact, since $A \succ q_1 \succ q_2$ implies $A \succ q_2$ we have $A \succ q \in I^{\mathfrak{F}}$ implies $A \succ I$.

An ideal I is said to be *of finite character* if $A \succ I$ implies there exists a character manifold B such that $A \subset B \succ I$.

An ideal I is said to be *of bounded character* if there is a basis B of I and $c > 0$ such that every $q \in B$ is of character c.

THEOREM 6.3. *If I is of bounded character, then I is of finite character.*

Proof. If $A \succ I$ and I is of bounded character, then there is a basis B of I and $c > 0$ such that every $q \in B$ is of character c. Consider the character ($2c$) hull of A, $\bar{A}^{2c}$. Then, by Theorem 2.7, $A \succ q$ and q of character c implies that $\bar{A}^{2c} \succ q$. But for any $q_1 \in I$ there is $q_2 \in B$ such that $q_1 \prec q_2$. Then $\bar{A}^{2c} \succ q_2 \succ q_1$ implies $\bar{A}^{2c} \succ q_1$. Therefore, since q_1 was arbitrary $\bar{A}^{2c} \succ I$.

If I_1 and I_2 are ideals, we say that I_1 *is equivalent to* I_2 and write $I_1 \sim I_2$ if, for every character manifold A, $A \succ I_1$ if and only if $A \succ I_2$.

THEOREM 6.4. *If I_1 and I_2 are ideals, then $I_1 \sim I_2$ if and only if; for every finite character quasi-norm q, $q \in I_1^{\mathfrak{F}}$ if and only if $q \in I_2^{\mathfrak{F}}$.*

Proof. If $I_1 \sim I_2$ and q is of finite character and such that $q \in I_1^{\mathfrak{F}}$ then there is $a > 0$ such that $A = \{x : q(x) < a\}$ is of finite character and furthermore $A \succ q$. Now since $q \in I_1^{\mathfrak{F}}$ we see that $A \succ I_1$ and since $I_1 \sim I_2$ we have $A \succ I_2$. Given any $q_1 \in I_2$ and $\varepsilon > 0$ there is a positive integer n such that $q_1((1/n)A) < \varepsilon$. Let $\delta = a/n$. Then $q(x) < \delta$ implies $q(nx) \leqq nq(x) \leqq n\delta = a$. Thus, $q(x) < \delta$ implies $nx \in A$ which means $x \in (1/n)A$ and therefore $q_1(x) < \varepsilon$. Thus, $q_1 \prec q$ and since q_1 was arbitrary $q \in I_2^{\mathfrak{F}}$.

For the converse, let A be of finite character and let q_A be the quasi-norm associated with A. We see that $q_A \in I_1^{\mathfrak{F}}$ and q_A is of finite character. Therefore, by assumption $q_A \in I_2^{\mathfrak{F}}$. But then $A \succ q_A$ gives $A \succ I_2$.

Given any ideal I_0 consider $\bar{I}_0 = \bigvee_{I \sim I_0} I$. $\bar{I}_0$ is called the *equivalent hull of* I_0.

THEOREM 6.5. *If* $M = \{q : q$ *is of finite character and* $q \in I_0^{\mathfrak{F}}\}$ *then* $\bar{I}_0 = M^{\mathfrak{I}}$.

Proof. Clearly M is never empty since q^* defined in Example 1.1 is of finite character. Theorem 6.4 shows that if $I \sim I_0$, then $M \subset I^{\mathfrak{F}}$. Therefore, $M^{\mathfrak{I}} \supset I^{\mathfrak{F}\mathfrak{I}} \supset I$. Hence, $I \sim I_0$ implies $I \subset M^{\mathfrak{I}}$.

Now suppose I_1 is such that $I \sim I_0$ implies that $I \subset I_1$. For any $q \in M^{\mathfrak{I}}$ consider $(\{q\} \cup I_0)^{\mathfrak{F}\mathfrak{I}}$. Then, if q_1 is of finite character and $q_1 \in I_0^{\mathfrak{F}}$ we see that $q_1 \succ q$ since $q \in M^{\mathfrak{I}}$. Therefore, $q_1 \in (\{q\} \cup I_0)^{\mathfrak{F}} = (\{q\} \cup I_0)^{\mathfrak{F}\mathfrak{I}\mathfrak{F}}$. Conversely, if q_2 is of finite character and $q_2 \in (\{q\} \cup I_0)^{\mathfrak{F}\mathfrak{I}\mathfrak{F}}$, then $(\{q\} \cup I_0)^{\mathfrak{F}\mathfrak{I}\mathfrak{F}} = (\{q\} \cup I_0)^{\mathfrak{F}} \subset I_0^{\mathfrak{F}}$ and we have $q_2 \in I^{\mathfrak{F}}$. Thus, we see that $(\{q\} \cup I_0)^{\mathfrak{F}\mathfrak{I}} \sim I_0$. Therefore, $I_1 \supset (\{q\} \cup I_0)^{\mathfrak{I}\mathfrak{F}} \ni q$. Hence, $q \in M^{\mathfrak{I}}$ implies $q \in I_1$ and we have $M^{\mathfrak{I}} \subset I_1$. Therefore $M^{\mathfrak{I}} = \bigvee_{I \sim I_0} I$.

COROLLARY. $\bar{I}_0$ *is reflexive.*

Proof. $\bar{I}_0^{\mathfrak{F}\mathfrak{I}} = M^{\mathfrak{I}\mathfrak{F}\mathfrak{I}} = M^{\mathfrak{I}} = \bar{I}_0$.

An ideal I is *simple* if $I \ni q_0 \succ q$ for all $q \in I$.

THEOREM 6.6. *An ideal* I is simple *if and only if* $I \cap I^{\mathfrak{F}} \neq \varnothing$.

Proof. If I is simple $I \ni q_0 \succ q$ for all $q \in I$ implies that $q_0 \in I \cap I^{\mathfrak{F}}$.
If $q_0 \in I \cap I^{\mathfrak{F}}$ then $I \ni q_0 \succ q$ for all $q \in I$ and I is simple.

7. **Relative ideals.** Let R be a linear manifold contained in S. Given a quasi-norm q on S we define a function q^R on R by setting $q^R(x) = q(x)$ for $x \in R$. It can easily be shown that q^R is a quasi-norm on the linear space R. It is called the *relative quasi-norm of* q *on* R. For any system B of quasi-norms on S, we let $B^R = \{q^R : q \in B\}$. The following properties are immediate consequences of the definition.

(1) If q is proper, then q^R is proper.

(2) If q is of finite character χ then q^R is of character χ.

(3) $q_1 \prec q_2$ implies $q_1^R \prec q_2^R$.

Let p be a quasi-norm on the linear space R. If we set

$$p^\infty(x) = \begin{cases} p(x) & \text{for } x \in R, \\ +\infty & \text{for } x \,\bar{\in}\, R, \end{cases}$$

we obtain a quasi-norm p^∞ on S which will be called the maximum extension of p over S. For any system B of quasi-norms on R we let $B^\infty = \{p^\infty : p \in B\}$.

THEOREM 7.1. *Let* q *be a quasi-norm on* S *and let* p *be a quasi-norm on* R. *Then* $q^R \prec p$ *implies* $q \prec p^\infty$.

Proof. For any $\varepsilon > 0$ there is $\delta > 0$ such that $p(x) < \delta$ implies $q^R(x) < \varepsilon$ for $x \in R$. But $p^\infty(x) < \delta$ implies $x \in R$ and hence $p(x) = p^\infty(x) < \delta$ gives us $q(x) = q^R(x) < \varepsilon$.

THEOREM 7.2. *If I is an ideal on S, then $I^R = \{q^R : q \in I\}$ is the basis of an ideal $I^{(R)}$ on R. This ideal is called the relative ideal of I on R. If F is a filter on S, then F^R is a filter on R called the relative filter of F on R.*

Proof. For a sequence $q_\nu^R \in I^R$ $(\nu = 1,2,\cdots)$ we see that $I \ni q_\nu$ $(\nu = 1,2,\cdots)$ and therefore there is $q_0 \in I$ such that $q_0 \succ q_\nu$ $(\nu = 1,2,\cdots)$. Thus, $I^R \ni q_0^R \succ q_\nu^R$ $(\nu = 1,2,\cdots)$. This shows that I^R is a basis for an ideal on R.

If F is a filter on S and p is a quasi-norm on R such that $p \succ q^R$ for some $q \in F$, then $p^\infty \succ q$ which implies $p^\infty \in F$. But then $p = (p^\infty)^R \in F^R$.

Given a system B of quasi-norms remember that $B^{\mathfrak{J}}(B^{\mathfrak{F}})$ is the ideal (filter) associated with this system.

THEOREM 7.3. *For a system B of quasi-norms on S we have $(B^R)^{\mathfrak{F}} = (B^{\mathfrak{F}})^R$ and $(B^{\mathfrak{J}})^{(R)} \subset (B^R)^{\mathfrak{J}}$.*

Proof. If $p \in (B^{\mathfrak{F}})^R$, then there is $q \in B^{\mathfrak{F}}$ such that $p = q^R$. But $q \in B^{\mathfrak{F}}$ implies that $q \succ q_1$ for all $q_1 \in B$ and hence $p = q^R \succ q_1^R$ for all $q_1 \in B$ which is the same as saying $p \in (B^R)^{\mathfrak{F}}$.

Conversely, if $p \in (B^R)^{\mathfrak{F}}$ then $p \succ q^R$ for all $q \in B$ and consequently $p^\infty \succ q$ for all $q \in B$. Thus, $p^\infty \in B^{\mathfrak{F}}$ implies $p = (p^\infty)^R \in (B^{\mathfrak{F}})^R$.

If $p \in (B^{\mathfrak{J}})^{(R)}$, then there is $q \in B^{\mathfrak{J}}$ such that $p \prec q^R$. Since $q \prec q_1$ for all $q_1 \in B$ we have $p \prec q^R \prec q_1^R$ for all $q_1 \in B$ which implies $p \in (B^R)^{\mathfrak{J}}$.

THEOREM 7.4. *If a filter F is reflexive, then the relative filter F^R is also reflexive.*

Proof. If $F = F^{\mathfrak{J}\mathfrak{F}}$ then we have $F^R = (F^{\mathfrak{J}\mathfrak{F}})^R = [(F^{\mathfrak{J}})^R]^{\mathfrak{F}}$. Since $[(F^{\mathfrak{J}})^R]^{\mathfrak{F}}$ is reflexive by Theorem 6.1, we see that F^R is reflexive.

We now consider a generalization of the inductive limit defined in [1].

If $R_\lambda (\lambda \in \Lambda)$ is a system of linear manifolds contained in S such that $S = \bigcup_{\lambda \in \Lambda} R_\lambda$ and such that, for any $\lambda_1, \lambda_2 \in \Lambda$ there is $\lambda_3 \in \Lambda$ such that $R_{\lambda_1} \cup R_{\lambda_2} \subset R_{\lambda_3}$, then we write $R_\lambda \uparrow_{\lambda \in \Lambda} S$ and say that the system $R_\lambda (\lambda \in \Lambda)$ increases to S.

THEOREM 7.5. *If $R_\lambda \uparrow_{\lambda \in \Lambda} S$ and I_λ is a reflexive ideal on R_λ for all $\lambda \in \Lambda$ then $\{q : q^{R_\lambda} \in I_\lambda \text{ for all } \lambda \in \Lambda\} = (\bigcup_{\lambda \in \Lambda} (I_\lambda^{\mathfrak{F}})^\infty)^{\mathfrak{J}}$.*

Proof. Let $I_0 = \{q : q^{R_\lambda} \in I_\lambda \text{ for all } \lambda \in \Lambda\}$. Then, $q \in I_0$ implies $q^{R_\lambda} \in I$ for all $\lambda \in \Lambda$. Hence $q^{R_\lambda} \prec p$ for all $p \in I_\lambda^{\mathfrak{F}}$ and therefore $q \prec p^\infty$ for all $p \in I_\lambda^{\mathfrak{F}}$. Since this is true for all $\lambda \in \Lambda$ we have $q \in (\bigcup_{\lambda \in \Lambda} (I_\lambda^{\mathfrak{F}})^\infty)^{\mathfrak{J}}$. Conversely, if $q \in (\bigcup_{\lambda \in \Lambda} (I_\lambda^{\mathfrak{F}})^\infty)^{\mathfrak{J}}$, then $q^{R_\lambda} \prec p^{R_\lambda}$ for all $p \in (I_\lambda^{\mathfrak{F}})^\infty$ and all $\lambda \in \Lambda$. But $p \in I_\lambda^{\mathfrak{F}}$ if and only if $p = (p^\infty)^{R_\lambda}$ where $p^\infty \in (I_\lambda^{\mathfrak{F}})^\infty$. Therefore $I_\lambda^{\mathfrak{F}} = ((I_\lambda^{\mathfrak{F}})^\infty)^{R_\lambda}$ for all $\lambda \in \Lambda$ and hence $q^{R_\lambda} \in I_\lambda^{\mathfrak{F}\mathfrak{J}} = I_\lambda$. Thus, $q \in I_0$ and we have $I_0 = (\bigcup_{\lambda \in \Lambda} (I_\lambda^{\mathfrak{F}})^\infty)^{\mathfrak{J}}$.

Now suppose $R_\lambda \uparrow_{\lambda\in\Lambda} S$ and $\boldsymbol{I}_\lambda$ is an ideal on R_λ for all $\lambda\in\Lambda$. The *upper limit* of $\boldsymbol{I}_\lambda(\lambda\in\Lambda)$ (written $\limsup_{\lambda\in\Lambda}\boldsymbol{I}_\lambda$) is defined by: $\limsup_{\lambda\in\Lambda}\boldsymbol{I}_\lambda = \bigvee\{\boldsymbol{I}: \forall_{\lambda_0\in\Lambda}\ \exists\lambda\in\Lambda$ such that $R_{\lambda_0}\subset R_\lambda$ and $\boldsymbol{I}^{R_\lambda}\subset \boldsymbol{I}_\lambda\}$. The *lower limit* of $\boldsymbol{I}_\lambda$ $(\lambda\in\Lambda)$ is defined by $\liminf_{\lambda\in\Lambda}\boldsymbol{I}_\lambda = \bigvee\{\boldsymbol{I}$: there is $\lambda_0\in\Lambda$ such that $R_\lambda\supset R_{\lambda_0}$ implies $\boldsymbol{I}^{R_\lambda}\subset\boldsymbol{I}_\lambda\}$. Since $R_\lambda(\lambda\in\Lambda)$ is an increasing system it is clear that $\liminf_{\lambda\in\Lambda}\boldsymbol{I}_\lambda\subset\limsup_{\lambda\in\Lambda}\boldsymbol{I}_\lambda$. If $\liminf_{\lambda\in\Lambda}\boldsymbol{I}_\lambda=\limsup_{\lambda\in\Lambda}\boldsymbol{I}_\lambda$ then we say that the system $\boldsymbol{I}_\lambda(\lambda\in\Lambda)$ is *convergent* to $\lim_{\lambda\in\Lambda}\boldsymbol{I}_\lambda=\limsup_{\lambda\in\Lambda}\boldsymbol{I}_\lambda=\liminf_{\lambda\in\Lambda}\boldsymbol{I}_\lambda$.

THEOREM 7.6. *If $R_\lambda\supset R_\rho$ implies that $\boldsymbol{I}_\lambda^{R_\rho}\subset\boldsymbol{I}_\rho$ (which implies $\boldsymbol{I}_\lambda^{(R_\rho)}\subset\boldsymbol{I}_\rho$) then $\boldsymbol{I}_\lambda(\lambda\in\Lambda)$ is convergent and* $\lim_{\lambda\in\Lambda}\boldsymbol{I}_\lambda=\{q:q^{R_\lambda}\in\boldsymbol{I}_\lambda$ *for all* $\lambda\in\Lambda\}$. *Furthermore, if for any $\lambda_0\in\Lambda$ there is $\lambda\in\Lambda$ such that $R_{\lambda_0}\subset R_\lambda$ and $\boldsymbol{I}_\lambda$ is reflexive, then* $\lim_{\lambda\in\Lambda}\boldsymbol{I}_\lambda$ *is also reflexive.*

Proof. Let $\boldsymbol{I}$ be an ideal such that for any $\lambda_0\in\Lambda$ there is $\lambda\in\Lambda$ such that $R_{\lambda_0}\subset R_\lambda$ and $\boldsymbol{I}^{R_\lambda}\subset\boldsymbol{I}_\lambda$. Then, if $q\in\boldsymbol{I}$ for any $\lambda_0\in\Lambda$ there is $\lambda\in\Lambda$ such that $q^{R_\lambda}\in\boldsymbol{I}_\lambda$ and $R_\lambda\supset R_{\lambda_0}$. By hypothesis $q^{R_{\lambda_0}}=(q^{R_\lambda})^{R_{\lambda_0}}\in\boldsymbol{I}_\lambda^{R_{\lambda_0}}\subset\boldsymbol{I}_{\lambda_0}$. Therefore, $q\in\{q:q^{R_\lambda}\in\boldsymbol{I}_\lambda$ for all $\lambda\in\Lambda\}$. (Call this last set $\boldsymbol{I}_0$.) Then $\boldsymbol{I}\subset\boldsymbol{I}_0$ and this shows that $\limsup_{\lambda\in\Lambda}\boldsymbol{I}_\lambda\subset\boldsymbol{I}_0$.

Now suppose $q\in\boldsymbol{I}_0$. Then, for any $\lambda_0\in\Lambda$ we have $\boldsymbol{I}_0^{R_\lambda}\subset\boldsymbol{I}_\lambda$ for $R_\lambda\supset R_{\lambda_0}$. Therefore $\boldsymbol{I}_0\subset\liminf_{\lambda\in\Lambda}\boldsymbol{I}_\lambda$. Thus, we see that $\boldsymbol{I}_\lambda(\lambda\in\Lambda)$ is convergent and $\lim_{\lambda\in\Lambda}\boldsymbol{I}_\lambda=\boldsymbol{I}_0$.

The second part of the theorem corresponds to the assumption that there is a system λ_ρ $(\rho\in\mathrm{P})$ such that, for any $\lambda\in\Lambda$, there is $\rho\in\mathrm{P}$ such that $R_\lambda\subset R_{\lambda_\rho}$ and $\boldsymbol{I}_{\lambda_\rho}$ is reflexive. Since $R_\lambda\subset R_{\lambda_\rho}$ implies $q^{R_\lambda}=(q^{R_{\lambda_\rho}})^{R_\lambda}$ and since $\boldsymbol{I}_{\lambda_\rho}^{R_\lambda}\subset\boldsymbol{I}_\lambda$ we see that $\lim_{\lambda\in\Lambda}\boldsymbol{I}_\lambda=\{q:q^{R_\lambda}\in\boldsymbol{I}_\lambda$ for all $\lambda\in\Lambda\}=\{q:q^{R_{P_\rho}}\in\boldsymbol{I}_{\lambda_\rho}$ for all $\rho\in\mathrm{P}\}$. Then, by Theorem 7.5 we have that $\lim_{\lambda\in\Lambda}\boldsymbol{I}_\lambda=(\bigcup_{\rho\in P}(\boldsymbol{I}_{\lambda_\rho}^{\mathfrak{F}})^{\infty})^{\mathfrak{J}}$ which is reflexive.

Now suppose $R_\lambda\uparrow_{\lambda\in\Lambda}S$ and $\boldsymbol{F}_\lambda$ is a filter on R_λ for all $\lambda\in\Lambda$. The *upper limit* of $\boldsymbol{F}_\lambda(\lambda\in\Lambda)$ is $\limsup_{\lambda\in\Lambda}\boldsymbol{F}_\lambda=\{p$: for all $\lambda_0\in\Lambda$ there is $\lambda\in\Lambda$ such that $R_{\lambda_0}\subset R_\lambda$ and $p^{R_\lambda}\in\boldsymbol{F}_\lambda\}$. The *lower limit* is defined to be: $\liminf_{\lambda\in\Lambda}\boldsymbol{F}_\lambda=\{p$: there is $\lambda_0\in\Lambda$ such that $p^{R_\lambda}\in\boldsymbol{F}_\lambda$ for all $\lambda\in\Lambda$ such that $R_\lambda\supset R_{\lambda_0}\}$. It is clear that both the upper limit and lower limit are filters such that $\liminf_{\lambda\in\Lambda}\boldsymbol{F}_\lambda\subset\limsup_{\lambda\in\Lambda}\boldsymbol{F}_\lambda$. If $\limsup_{\lambda\in\Lambda}\boldsymbol{F}_\lambda=\liminf_{\lambda\in\Lambda}\boldsymbol{F}_\lambda$, then $\boldsymbol{F}_\lambda(\lambda\in\Lambda)$ is said to be *convergent* to $\lim_{\lambda\in\Lambda}\boldsymbol{F}_\lambda$.

THEOREM 7.7. *If $R_\lambda\supset R_\rho$ implies $\boldsymbol{F}_\lambda^{R_\rho}\subset\Lambda_\rho$, then $\boldsymbol{F}_\lambda(\lambda\in\Lambda)$ is convergent and* $\lim_{\lambda\in\Lambda}\boldsymbol{F}_\lambda=\{p:p^{R_\lambda}\in\boldsymbol{F}_\lambda$ *for all* $\lambda\in\Lambda\}$.

Proof. In any case it is clear that

$$\{p:p^{R_\lambda}\in\boldsymbol{F}_\lambda \text{ for all } \lambda\in\Lambda\}\ \subset\ \liminf_{\lambda\in\Lambda}\boldsymbol{F}_\lambda\subset\limsup_{\lambda\in\Lambda}\boldsymbol{F}_\lambda.$$

Suppose $p\in\limsup_{\lambda\in\Lambda}\boldsymbol{F}_\lambda$. Then, for any $\lambda_0\in\Lambda$ there is $\lambda\in\Lambda$ such that $R_{\lambda_0}\subset R_\lambda$ and $p^{R_\lambda}\in\boldsymbol{F}_\lambda$. By assumption $\boldsymbol{F}_\lambda^{R_{\lambda_0}}\subset\boldsymbol{F}_{\lambda_0}$. Therefore $p^{R_{\lambda_0}}=(p^{R_\lambda})^{R_{\lambda_0}}\in\boldsymbol{F}_\lambda^{R_{\lambda_0}}\in\boldsymbol{F}_{\lambda_0}$. This gives us $\limsup_{\lambda\in\Lambda}\boldsymbol{F}_\lambda\subset\{p:p^{R_\lambda}\in\boldsymbol{F}_\lambda$ for all $\lambda\in\Lambda\}$.

THEOREM 7.8. *If $\boldsymbol{F}_\lambda^{R_\rho}=\boldsymbol{F}_\rho$ for $R_\rho\subset R_\lambda$, then, setting* $\lim_{\lambda\in\Lambda}\boldsymbol{F}_\lambda=\boldsymbol{F}_0$, *we have $\boldsymbol{F}_0^{R_\lambda}=\boldsymbol{F}_\lambda$ for $\lambda\in\Lambda$.*

Proof. From the preceding theorem we know that $\lim_{\lambda \in \Lambda} \boldsymbol{F}_\lambda = \{p : p^{R_\lambda} \in \boldsymbol{F}_\lambda$ for all $\lambda \in \Lambda\}$. From this it is clear that $\boldsymbol{F}_0^{R_\lambda} \subset \boldsymbol{F}_\lambda$ for all $\lambda \in \Lambda$. If $p \in \boldsymbol{F}_\lambda$, then $p = (p^\infty)^{R_\lambda}$ and we need only show that $p^\infty \in \boldsymbol{F}_0$. For any $\lambda_1 \in \Lambda$ there is $\lambda_2 \in \Lambda$ such that $R_\lambda \cup R_{\lambda_1} \subset R_{\lambda_2}$. Then by assumption, there is $q \in \boldsymbol{F}_{\lambda_2}$ such that $q^{R_\lambda} = p$. Then $(p^\infty)^{R_{\lambda_2}} \succ q$ implies that $(p^\infty)^{R_{\lambda_2}} \in \boldsymbol{F}_{\lambda_2}$. But then we see that

$$(p^\infty)^{R_{\lambda_1}} = ((p^\infty)^{R_{\lambda 2}})^{R_{\lambda_1}} \in \boldsymbol{F}_{\lambda_2}^{R_{\lambda_1}} = \boldsymbol{F}_{\lambda_1}.$$

Thus, $p^\infty \in \boldsymbol{F}_0$ and we have $\boldsymbol{F}_0^{R_\lambda} = \boldsymbol{F}_\lambda$.

An ideal $\boldsymbol{I}$ on S is said to be *compatible* to $R_\lambda \uparrow_{\lambda \in \Lambda} S$ if $\lim_{\lambda \in \Lambda} \boldsymbol{I}^{(R_\lambda)} = \boldsymbol{I}$. Note that the required limit always exists by Theorem 7.6 since $(\boldsymbol{I}^{R_\lambda})^{R_\rho} = \boldsymbol{I}^{R_\rho}$ for $R_\rho \subset R_\lambda$ (which gives $(\boldsymbol{I}^{(R_\lambda)})^{(R_\rho)} = \boldsymbol{I}^{(R_\rho)}$). Also we have $\lim_{\lambda \in \Lambda} \boldsymbol{I}^{(R_\lambda} = \{q : q^{R}_{\lambda} \in \boldsymbol{I}^{R_\lambda}\} \supset \boldsymbol{I}$ in any case.

THEOREM 7.9. *If $\boldsymbol{I}_\lambda^{R_\rho} \subset \boldsymbol{I}_\rho$ for $R_\rho \subset R_\lambda$, then $\lim_{\lambda \in \Lambda} \boldsymbol{I}_\lambda$ is an ideal which is compatible to $R_\lambda \uparrow_{\lambda \in \Lambda} S$.*

Proof. Let $\boldsymbol{I}_0 = \lim_{\lambda \in \Lambda} \boldsymbol{I}_\lambda = \{q : q^{R_\lambda} \in \boldsymbol{I}_\lambda$ for all $\lambda \in \Lambda\}$. Then clearly $\boldsymbol{I}_0^{R_\lambda} \subset \boldsymbol{I}_\lambda$ for all $\lambda \in \Lambda$. Then $\lim_{\lambda \in \Lambda} \boldsymbol{I}_0^{(R_\lambda)} = \{q : q^{R_\lambda} \in \boldsymbol{I}_0^{(R_\lambda)}$for all $\lambda \in \Lambda\} \subset \{q : q^{R_\lambda} \in \boldsymbol{I}_\lambda$ for all $\lambda \in \Lambda\} = \boldsymbol{I}_0$. Therefore $\lim_{\lambda \in \Lambda} \boldsymbol{I}_0^{(R_\lambda)} \subset \boldsymbol{I}_0$ and the reverse inclusion is always true.

Given an ideal $\boldsymbol{I}$ on S and a system $R_\lambda \uparrow_{\lambda \in \Lambda} S$ suppose there is an ideal $\hat{\boldsymbol{I}}$ on S such that: (1) $\hat{\boldsymbol{I}} \supset \boldsymbol{I}$; (2) $\hat{\boldsymbol{I}}$ is compatible to $R_\lambda \uparrow_{\lambda \in \Lambda} S$; and (3) $\boldsymbol{I}_1 \supset \boldsymbol{I}$ and $\boldsymbol{I}_1$ compatible to $R_\lambda \uparrow_{\lambda \in \Lambda} S$ implies $\hat{\boldsymbol{I}} \subset \boldsymbol{I}_1$. Such an ideal is called the *compatible hull* of $\boldsymbol{I}$.

THEOREM 7.10. *For any ideal $\boldsymbol{I}$ the compatible hull $\hat{\boldsymbol{I}}$ exists and is given by $\hat{\boldsymbol{I}} = \lim_{\lambda \in \Lambda} \boldsymbol{I}^{(R_\lambda)}$.*

Proof. The remark before Theorem 7.9 shows that $\boldsymbol{I} \subset \lim_{\lambda \in \Lambda} \boldsymbol{I}^{(R_\lambda)}$ and Theorem 7.9 shows that $\lim_{\lambda \in \Lambda} \boldsymbol{I}^{(R_\lambda)}$ is compatible. If $\boldsymbol{I} \subset \boldsymbol{I}_1$ is compatible, then we have $\boldsymbol{I}^{R_\lambda} \subset \boldsymbol{I}_1^{R_\lambda}$ and therefore:

$$\begin{aligned}\lim_{\lambda \in \Lambda} \boldsymbol{I}^{(R_\lambda)} &= \{q : q^{R_\lambda} \in \boldsymbol{I}^{R_\lambda} \forall \lambda \in \Lambda\} \subset \{q : q^{R_\lambda} \in \boldsymbol{I}_1^{R_\lambda} \text{ for all } \lambda \in \Lambda\} \\ &= \lim_{\lambda \in \Lambda} \boldsymbol{I}_1^{(R_\lambda)} = \boldsymbol{I}_1.\end{aligned}$$

A filter $\boldsymbol{F}$ is said to be *compatible* to $R_\lambda \uparrow_{\lambda \in \Lambda} S$ if $\lim_{\lambda \in \Lambda} \boldsymbol{F}^{R_\lambda} = \boldsymbol{F}$. As in the case of ideals, this limit always exists and we have $\lim_{\lambda \in \Lambda} \boldsymbol{F}^{R_\lambda} = \{p : p^{R_\lambda} \in \boldsymbol{F}^{R}$ for all $\lambda \in \Lambda\} \supset \boldsymbol{F}$.

THEOREM 7.11. *If $\boldsymbol{F}_\lambda^{R_\rho} \subset \boldsymbol{F}_\rho$ for $R_\rho \subset R_\lambda$, then $\lim_{\lambda \in \Lambda} \boldsymbol{F}_\lambda$ is compatible to $R_\lambda \uparrow_{\lambda \in \Lambda} S$.*

Proof. Let $\boldsymbol{F}_0 = \lim_{\lambda \in \Lambda} \boldsymbol{F}_\lambda = \{p : p^{R_\lambda} \in \boldsymbol{F}_\lambda$ for all $\lambda \in \Lambda\}$. Then $\boldsymbol{F}_0^{R_\lambda} \subset \boldsymbol{F}_\lambda$ for all $\lambda \in \Lambda$ and we have: $\lim_{\lambda \in \Lambda} \boldsymbol{F}_0^{R_\lambda} = \{p : p^{R_\lambda} \in \boldsymbol{F}_0^{R_\lambda}$ for all $\lambda \in \Lambda\} \subset \{p : p^{R_\lambda} \in \boldsymbol{F}_\lambda$ for all $\lambda \in \Lambda\} = \boldsymbol{F}_0$. Therefore $\lim_{\lambda \in \Lambda} \boldsymbol{F}_0^{R} = \boldsymbol{F}_0$.

Given a filter F on S and a system $R_\lambda \uparrow_{\lambda \in \Lambda} S$ suppose there is a filter $\hat{F}$ such that (1) $\hat{F} \supset F$; (2) $\hat{F}$ is compatible to $R_\lambda \uparrow_{\lambda \in \Lambda} S$; and (3) $F \subset F_1$ compatible implies $\hat{F} \subset F_1$. Such a filter $\hat{F}$ is called the *compatible hull* of F.

THEOREM 7.12. *For any filter F the compatible hull $\hat{F}$ exists and is given by $\hat{F} = \lim_{\lambda \in \Lambda} F^{R_\lambda}$.*

Proof. Previous results show that (1) and (2) are satisfied and we need only prove (3). If $F \subset F_1$ compatible, then we have $F^{R_\lambda} \subset F_1^{R_\lambda}$ for all $\lambda \in \Lambda$.

$$\lim_{\lambda \in \Lambda} F^{R_\lambda} = \{q : q^{R_\lambda} \in F^{R_\lambda} \text{ for all } \lambda \in \Lambda\} \supset \{q : q^{R_\lambda} \in F_1^{R_\lambda} \text{ for all } \lambda \in \Lambda\}$$
$$= \lim_{\lambda \in \Lambda} F_1^{R_\lambda} = F_1.$$

THEOREM 7.13. *If $R_\rho \subset R_\lambda$ implies $F_\lambda^{R_\rho} = F_\rho$, then*

$$\left(\bigcup_{\lambda \in \Lambda} F_\lambda^\infty\right)^\wedge = \{p : p^{R_\lambda} \in F_\lambda \textit{ for all } \lambda \in \Lambda\} = \lim_{\lambda \in \Lambda} F_\lambda.$$

Proof. We first show that $(\bigcup_{\lambda \in \Lambda} F_\lambda^\infty)^{R_\lambda} = F_\lambda$ for all $\lambda \in \Lambda$. If $p \in \bigcup_{\lambda \in \Lambda} F_\lambda^\infty$, then there is $\lambda_0 \in \Lambda$ such that $p = q^\infty$ for some $q \in F_{\lambda_0}$. For any other $\lambda_1 \in \Lambda$ there is $\lambda_2 \in \Lambda$ such that $R_{\lambda_0} \cup R_{\lambda_1} \subset R_{\lambda_2}$. Then, since $F_{\lambda_2}^{R_{\lambda_0}} = F_{\lambda_0}$, $p^{R_{\lambda_2}} \in F_{\lambda_2}$ and therefore $p^{R_{\lambda_1}} = (p^{R_{\lambda_2}})^{R_{\lambda_1}} \in F_{\lambda_2}^{R_{\lambda_1}} = F_{\lambda_1}$. Thus, we see that $(\bigcup_{\lambda \in \Lambda} F_\lambda^\infty)^{R_\lambda} \subset F_\lambda$. The opposite inclusion follows from the fact that $F_\lambda = (F_\lambda^\infty)^{R_\lambda}$. By the previous theorem we have $(\bigcup_{\lambda \in \Lambda} F_\lambda^\infty)^\wedge = \lim_{\lambda \in \Lambda} (\bigcup_{\lambda \in \Lambda} F_\lambda^\infty)^{R_\lambda} = \lim_{\lambda \in \Lambda} F_\lambda = \{p : p^{R_\lambda} \in F_\lambda \text{ for all } \lambda \in \Lambda\}$.

THEOREM 7.14. *For any ideal I on S and any system $R_\lambda \uparrow_{\lambda \in \Lambda} S$ we have $\hat{I}^{\mathfrak{F}} \subset (\hat{I}^{\mathfrak{F}})^\wedge$.*

Proof. $q \in \hat{I}^{\mathfrak{F}}$ implies $q \in I^{\mathfrak{F}}$ since $\hat{I} \supset I$. Consequently $q^{R_\lambda} \in I^{\mathfrak{F} R_\lambda}$ for all $\lambda \in \Lambda$. Then $q \in \{p : p^{R_\lambda} \in I^{\mathfrak{F} R_\lambda}$ for all $\lambda \in \Lambda\} = (I^{\mathfrak{F}})^\wedge$. Thus $\hat{I}^{\mathfrak{F}} \subset (I^{\mathfrak{F}})^\wedge$.

THEOREM 7.15. *For any system $R_\lambda \uparrow_{\lambda \in \Lambda} S$ we have $\hat{F}^{\mathfrak{I}} \subset \lim_{\lambda \in \Lambda} F^{R_\lambda \mathfrak{I}}$.*

Proof. $q \in \hat{F}^{\mathfrak{I}}$ implies $q^{R_\lambda} \in \hat{F}^{\mathfrak{I} R_\lambda} \subset \hat{F}^{R_\lambda \mathfrak{I}} = F^{R_\lambda \mathfrak{I}}$ for all $\lambda \in \Lambda$. (The last relation follows from the fact that: $F^{R_\lambda} \subset \hat{F}^{R_\lambda} = \{p : p^{R_\lambda} \in F^{R_\lambda}$ for all $\lambda \in \Lambda\}^{R_\lambda} \subset F^{R_\lambda}$). $F^{R_\lambda \mathfrak{I}}$ $(\lambda \in \Lambda)$ is a convergent system of ideals since $R_\lambda \subset R_\mu$ implies

$$(F^{R_\mu \mathfrak{I}})^{R_\lambda} \subset (F^{R_\mu})^{R_\lambda \mathfrak{I}} = F^{R_\lambda \mathfrak{I}} \text{ for all } \lambda \in \Lambda.$$

$q^{R_\lambda} \in F^{R_\lambda \mathfrak{I}}$ for all $\lambda \in \Lambda$ gives $q \in \{p : p^{R_\lambda} \in F^{R_\lambda \mathfrak{I}}$ for all $\lambda \in \Lambda\} = \lim_{\lambda \in \Lambda} F^{R_\lambda \mathfrak{I}}$. Therefore $\hat{F}^{\mathfrak{I}} \subset \lim_{\lambda \in \Lambda} F^{R_\lambda \mathfrak{I}}$.

8. Finite-dimensional spaces.

THEOREM 8.1. *If S is a finite-dimensional space and q and p are quasi-norms such that q is pure and proper and p is proper, then $q \succ p$.*

Proof. Choose a basis $a_1, a_2, \cdots, a_n$ for S and let $m(\sum_{\nu=1}^n \alpha_\nu a_\nu) = \max_{\nu=1,2,\dots,n} |\alpha_\nu|$. If we can show that q proper implies $m \succ q$ and that q proper and pure implies $q \succ m$, we will have proved the desired result.

Let q be proper. Then, given $\varepsilon > 0$, for each a_ν ($\nu = 1,2,\cdots,n$) there is $\delta_\nu > 0$ such that $q(\xi a_\nu) < \varepsilon/n$ for $|\xi| \leqq \delta_\nu$. Let $\delta = \min_{\nu=1,2,\ldots,n}\delta_\nu$. Then, when $m(\sum_{\nu=1}^n \alpha_\nu a_\nu) < \delta$, we have $q(\sum_{\nu=1}^n \alpha_\nu a_\nu) \leqq \sum_{\nu=1}^n q(\alpha_\nu a_\nu) \leqq \sum_{\nu=1}^n \varepsilon/n = \varepsilon$. Thus $m \succ q$.

Now we want to show that if q is proper and q does not dominate m, then q is not pure. If q is proper and q does not dominate m, then there is $\varepsilon > 0$ such that, for each $\delta > 0$ there is $x \in \{x : q(x) < \delta\}$ satisfying $m(x) \geqq \varepsilon$. For $\nu = 1,2,\cdots$ we can find $x_\nu \in \{x : q(x) < 1/\nu\}$ such that $m(x_\nu) \geqq \varepsilon$. By multiplying x_ν by a scalar of absolute value less than or equal to one, we can assume that $\varepsilon \leqq m(x_\nu) \leqq 2\varepsilon$ for $\nu = 1,2,\cdots$. If $x_\nu = \sum_{\mu=1}^n \alpha_{\nu,\mu} a_\mu$ this implies $|\alpha_{\nu,\mu}| \leqq 2\varepsilon$ for all $\nu = 1,2,\cdots$ and all $\mu = 1,2,\cdots,n$. We then have n bounded sequences of real numbers and we can choose a subsequence ν_ρ ($\rho = 1,2,\cdots$) such that they all converge: $\lim_{\rho\to\infty} \alpha_{\nu_\rho,\mu} = \beta_\mu$ ($\mu = 1,2,\cdots,n$). Let $x_0 = \sum_{\mu=1}^n \beta_\mu a_\mu$. Then we see that $m(x_{\nu_\rho} - x_0) \to 0$ since $m(x_{\nu_\rho} - x_0) = \max_{\mu=1,2,\ldots,n} |\alpha_{\nu_\rho} - \beta_\mu| \to 0$. The fact that $m(x_0) \geqq m(x_{\nu_\rho}) - m(x_{\nu_\rho} - x_0) \geqq \varepsilon - m(x_{\nu_\rho} - x_0)$ for all $\rho = 1,2,\cdots$ shows that $m(x_0) \geqq \varepsilon > 0$.

Since q is proper we know from the first part of the proof that $m \succ q$. Therefore $m(x_{\nu_\rho} - x_0) \to 0$ implies that $q(x_{\nu_\rho} - x_0) \to 0$. But then $q(x_0) \leqq q(x_{\nu_\rho}) + q(x_0 - x_{\nu_\rho})$ for all $\rho = 1,2,\cdots$. Since both of the last terms tend to zero, we see that $q(x_0) = 0$ which shows that q is not pure.

THEOREM 8.2. *If S is a linear space such that every pure and proper quasi-norm on S dominates every proper quasi-norm on S, then S is finite-dimensional.*

Proof. In light of Theorem 8.1 we need only show that if S is infinite-dimensional there is a pure proper quasi-norm on S which does *not* dominate every proper quasi-norm. Let $a_\alpha (\alpha \in A)$ be a Hamel basis for S. Define quasi-norms q and p by $q(\sum_{\alpha\in A} \nu_\alpha a_\alpha) = \max_{\alpha\in A} |\beta_\alpha|$ and $p(\sum_{\alpha\in A} \beta_\alpha a_\alpha) = \sum_{\alpha\in A} |\beta_\alpha|$. Then q is proper and pure but q does *not* dominate p. To see this choose $\varepsilon = 1$ and for any $\delta > 0$ choose n such that $n(\delta/2) > 1$. Then, let $\sum_{\alpha\in A} \nu_\alpha a_\alpha$ be such that $\nu_\alpha = \delta/2$ for n different values of α and such that $\nu_\alpha = 0$ for all other $\alpha \in A$. Then we have $q(\sum_{\alpha\in A} \nu_\alpha a_\alpha) = \delta/2 < \delta$ bu $p(\sum_{\alpha\in A} \nu_\alpha a_\alpha) = n(\delta/2) > 1$. Hence, q does not dominate p.

An ideal $\boldsymbol{I}$ is said to be *pure* if $x \neq 0$ implies that there is $q \in \boldsymbol{I}$ such that $q(x) \neq 0$.

THEOREM 8.3. *If S is finite-dimensional and $\boldsymbol{I}$ is pure, then $\boldsymbol{I}$ contains a quasi-norm q_0 which is pure.*

Proof. If S is one-dimensional and $\boldsymbol{I}$ is pure, then $x \neq 0$ implies there is $q_0 \in \boldsymbol{I}$ such that $q_0(x) \neq 0$. But then $(1/\nu) q_0(x) \leqq q_0((1/\nu)x)$ shows that $q_0(\alpha x) \neq 0$ for all $\alpha \neq 0$.

To complete the induction argument we assume the result holds when S is of dimension less than or equal to n. If S is $(n+1)$-dimensional, choose $0 \neq q_1 \in \boldsymbol{I}$ and let $R = \{x : q_1(x) = 0\}$. Since $q_1 \neq 0$ (dimension R) $\leqq n$ and the fact that $\boldsymbol{I}^{(R)}$

is pure on R shows that there is $p \in I^{(R)}$such that p is pure on R by the induction hypothesis. By definition of $I^{(R)}$ there is $q_2 \in I$ such that $q_2^R \succ p$ which implies q_2^R is pure on R. Now let $q_0 = (q_1 + q_2) \in I$. For any $0 \neq z \in S$ we have either $z \in R$ which implies

$$q_0(z) = q_2(z) = q_2^R(z) \neq 0$$

or $z \bar{\in} R$ which gives

$$q_0(z) \geqq q_1(z) \neq 0.$$

Therefore q_0 is pure.

THEOREM 8.4. *A linear space S is finite-dimensional if and only if there is exactly one pure, proper ideal on the space.*

Proof. If S is finite-dimensional and I is a pure, proper ideal on S then, by the previous theorem I contains a pure, proper quasi-norm q. By Theorem 8.1, if p is any proper quasi-norm on S we have $q \succ p$. Thus $\{q\}$ is a basis for I and I is uniquely defined as the strongest proper ideal on S.

If S is not finite-dimensional, then by Theorem 8.2 there is a pure, proper quasi-norm q and a proper quasi-norm p such that $p \bar{\in} \{q\}^{\mathfrak{I}}$. But then $p \in \{p + q\}^{\mathfrak{I}}$ and we see that $\{q\}^{\mathfrak{I}} \neq \{p + q\}^{\mathfrak{I}}$ even though both are pure, proper ideals.

THEOREM 8.5. *On a finite-dimensional space the unique pure, proper ideal has a basis $\{q\}$ where q is a quasi-norm of character* 1.

Proof. Let a_ν $(\nu = 1, 2, \cdots, n)$ be a basis for the space S and define q by:

$$q(x) = q\left(\sum_{\nu=1}^{n} \alpha_\nu a_\nu\right) = \max_{\nu = 1, \ldots, n} |\alpha_\nu|.$$

Then q is clearly a pure, proper quasi-norm and therefore $\{q\}^{\mathfrak{I}}$ is the unique, pure, proper ideal on S. Since $q(\frac{1}{2}x) = \max_{\nu = 1,2,\ldots,n} |\frac{1}{2}\alpha_\nu| = \frac{1}{2}q(x)$ we see that q is of character 1.

Now let S be a linear space and let S_λ $(\lambda \in \Lambda)$ be a system of finite-dimensional submanifolds of S such that $S_\lambda \uparrow_{\lambda \in \Lambda} S$.

THEOREM 8.6. *If I_1 and I_2 are pure, proper ideals on S then $I_1^{(S_\lambda)} = I_2^{(S_\lambda)}$ for all $\lambda \in \Lambda$.*

Proof. This follows from the previous theorem since $I_1^{(S_\lambda)}$ and $I_2^{(S_2}$ are pure, proper ideals.

THEOREM 8.7. *There is a unique, pure, proper ideal which is compatible to $S_\lambda \uparrow_{\lambda \in \Lambda} S$.* We denote this ideal by I_0.

Proof. If I_1 and I_2 are pure, proper and compatible to $S_\lambda \uparrow_{\lambda \in \Lambda} S$ then $I_1 = \lim_{\lambda \in \Lambda} I_1^{S_\lambda} = \lim_{\lambda \in \Lambda} I_2^{S_\lambda} = I_2$. This shows that any such ideal will be unique.

But there does exist such an ideal since, if I_λ is the unique, pure, proper ideal on S_λ, it is clear that $\lim_{\lambda\in\Lambda} I_\lambda$ is pure and proper and compatible with $S_\lambda \uparrow_{\lambda\in\Lambda} S$.

THEOREM 8.8. *For any pure, proper ideal I on S the compatible hull $\hat{I}$ of I equals I_0.*

Proof. We know $\hat{I} = \lim_{\lambda\in\Lambda} I^{S_\lambda} = \lim_{\lambda\in\Lambda} I_\lambda = I_0$. ($I_\lambda$ being the unique, pure, proper ideal on S_λ.)

Thus, we see that I_0 is the strongest pure, proper ideal. Since we can always find a system of finite-dimensional submanifolds S_λ ($\lambda \in \Lambda$) such that $S_\lambda \uparrow_{\lambda\in\Lambda} S$ we see that every linear space S has a strongest pure, proper ideal I_0. Let I_1 be the ideal of all proper quasi-norms on S. Then, since $I_1 \supset I_0$ we see that I_1 is pure and therefore $I_1 = I_0$.

9. **Monotone quasi-norms.** In the rest of the paper we will always assume that S is a linear lattice. We will make use of the notation $B_x = \{y \in S : |y| \leqq |x|\}$ for $x \in S$.

A quasi-norm q on S is said to be *semimonotone* if $B_x \succ q$ for all $x \in S$.

THEOREM 9.1. *If q is semimonotone, then q is proper.*

Proof. $x \in B_x$ implies $0 \leqq \lim_{a\to 0} q(ax) \leqq \lim_{a\to 0} q(aB_x) = 0$.

A quasi-norm q on S is said to be *monotone* if $|x| \leqq |y|$ implies $q(x) \leqq q(y)$.

THEOREM 9.2. *Let q be a quasi-norm on S. If we define a quasi-norm $\bar{q}$ by $\bar{q}(x) = q(B_x)$, we obtain a monotone quasi-norm. $\bar{q}$ is proper if and only if q is semimonotone.*

Proof. Clearly $\bar{q}(0) = 0$ and $0 \leqq \bar{q}(x)$ for all $x \in S$. If $|a| \leqq |b|$, then

$$\bar{q}(ax) = q(B_{ax}) \leqq q(B_{bx}) = \bar{q}(bx).$$

To prove the triangle inequality, we need the following lemma:

LEMMA. *$|z| \leqq |x| + |y|$ if and only if we can find z_1 and z_2 such that $z = z_1 + z_2$ while $|z_1| \leqq |x|$ and $|z_2| \leqq |y|$.*

Proof. $|z| \leqq |x| + |y|$ implies $0 \leqq |z| = z^+ + z^- \leqq |x| + |y|$ and hence $0 \leqq z^+ \leqq |x| + |y|$. We can then write $z^+ = a + b$ where $0 \leqq a \leqq |x|$ and $0 \leqq b \leqq |y|$. Similarly $z^- = c + d$ where $0 \leqq c \leqq |x|$ and $0 \leqq d \leqq |y|$. Then $z = z^+ - z^- = (a + b) - (c + d) = (a - c) + (b - d)$ where $-|x| \leqq a - c \leqq |x|$ and $-|y| \leqq b - d \leqq |y|$. Thus, $|a - c| \leqq |x|$ and $|b - d| \leqq |y|$ and the lemma is established.

$$\bar{q}(x + y) = q(B_{(x+y)}) = \sup_{|z| \leqq |x+y|} q(z) \leqq \sup_{|z| \leqq |x| + |y|} q(z)$$

$$= \sup_{|z_1| \leqq |x|;\, |z_2| \leqq |y|} q(z_1 + z_2) \leqq \sup_{|z_1| \leqq |x|;\, |z_2| \leqq |y|} \{q(z_1) + q(z_2)\} = \bar{q}(x) + \bar{q}(y)$$

Thus, we see that $\bar{q}$ is a quasi-norm. It is clear that $\bar{q}$ is monotone since if $|x| \leqq |y|$ then $B_x \subset B_y$ implies $\bar{q}(x) = q(B_x) \leqq q(B_y) = \bar{q}(y)$. It also follows easily that $\bar{q}$ is proper if and only if q is semimonotone.

THEOREM 9.3. *If q_1, q are quasi-norms such that q_1 is monotone and $q \prec q_1$, then $\bar{q} \prec q_1$.*

Proof. For any $\varepsilon > 0$ there is $\delta > 0$ such that $q_1(x) < \delta$ implies $q(x) < \varepsilon$. If $q_1(x) < \delta$ and $|y| \leqq |x|$ then $q_1(y) \leqq q_1(x) < \delta$ implies $q(y) < \varepsilon$. Therefore $\bar{q}(x) = q(B_x) = \sup_{|y| \leqq |x|} q(y) \leqq \varepsilon$.

THEOREM 9.4. *For a quasi-norm q on S we have $\bar{q} \prec q$ if and only if for any $\varepsilon > 0$ there is $\delta > 0$ such that $q(x) < \delta$ implies $q(B_x) < \varepsilon$.*

Proof. This is clear by definition of $\bar{q}$.

A quasi-norm q is called *sequentially continuous* if order-$\lim_{\nu \to \infty} x_\nu = 0$ implies $\lim_{\nu \to \infty} q(x_\nu) = 0$.

THEOREM 9.5. *If q, q_1 are quasi-norms such that q_1 is sequentially continuous and $q \prec q_1$, then q is sequentially continuous.*

Proof. Suppose order-$\lim_{\nu \to \infty} x_\nu = 0$. For $\varepsilon > 0$ there is $\delta > 0$ such that $q_1(x) < \delta$ implies $q(x) < \varepsilon$. For this δ there is ν_0 such that $q_1(x) < \delta$ for $\nu \geqq \nu_0$. But then $q(x_\nu) < \varepsilon$ for $\nu \geqq \nu_0$ and therefore $\lim_{\nu \to \infty} q(x_\nu) = 0$.

THEOREM 9.6. *If q is a sequentially continuous quasi-norm and S is Archimedean, then q is semimonotone.*

Proof. Suppose q is not semimonotone. Then there is $x \in S$ and $\varepsilon > 0$ such that $q((1/\nu) B_x) \geqq \varepsilon$ for all $\nu = 1, 2, \cdots$. Therefore, there is a sequence $x_\nu \in B_x$ $(\nu = 1, 2, \cdots)$ such that $q((1/\nu) x_\nu) > \varepsilon/2$ for all $\nu = 1, 2, \cdots$. But $x_\nu \in B_x$ implies that $|x_\nu| \leqq |x|$. Hence $|(1/\nu) x_\nu| \leqq |(1/\nu) x|$. This means order-$\lim_{\nu \to \infty}(1/\nu) x_\nu = 0$ since S is Archimedean. Therefore, we have $\lim_{\nu \to \infty} q((1/\nu) x_\nu) = 0$ which contradicts $q((1/\nu) x_\nu) > \varepsilon/2$ for all $\nu = 1, 2, \cdots$.

Since semimonotone implies proper, we see that every sequentially continuous quasi-norm on an Archimedean space is proper.

A quasi-norm q is *semicontinuous* if $|x_\lambda| \uparrow_{\lambda \in \Lambda} |x|$ implies that $q(x) = \sup_{\lambda \in \Lambda} q(x_\lambda)$. If we let $\Lambda = \{1, 2\}$ and consider $|x_1| \leqq |x_2|$, then $q(x_2) = \sup_{\lambda \in \{1,2\}} q(x_\lambda) \geqq q(x_1)$. Therefore, if a quasi-norm is semicontinuous, it is monotone.

A quasi-norm q is *monotone continuous* if it is monotone and $|x_\lambda| \downarrow_{\lambda \in \Lambda} 0$ implies $\inf_{\lambda \in \Lambda} q(x_\lambda) = 0$.

THEOREM 9.7. *If q is monotone continuous, then it is semicontinuous.*

Proof. Suppose $|x_\lambda| \uparrow_{\lambda \in \Lambda} |x|$. Then

$$|x| - |x_\lambda| = ||x| - |x_\lambda|| \downarrow_{\lambda \in \Lambda} 0.$$

Since q is monotone continuous $\inf_{\lambda\in\Lambda} q(|x|-|x_\lambda|)=0$. But

$$0 \leqq q(|x|)-q(|x_\lambda|) = q(x)-q(x_\lambda)$$

since q is monotone. Also, for any quasi-norm it is true that

$$|q(y)-q(z)| \leqq q(y-z).$$

Therefore $0 \leqq \inf_{\lambda\in\Lambda}\{q(x)-q(x_\lambda)\} \leqq \inf_{\lambda\in\Lambda} q(|x|-|x_\lambda|)=0$. Hence $0=\inf_{\lambda\in\Lambda}\{q(x)-q(x_\lambda)\}=q(x)-\sup_{\lambda\in\Lambda} q(x_\lambda)$.

THEOREM 9.8. *If a quasi-norm q is monotone continuous, then it is sequentially continuous.*

Proof. If order-$\lim_{\nu\to\infty} x_\nu = 0$, then, by definition, there is a sequence l_ν $(\nu=1,2,\cdots)$ such that $|x_\nu| \leqq l_\nu \downarrow_{(\nu=1,2,\ldots)} 0$. But then we have $0 \leqq q(x_\nu) \leqq q(l_\nu)\downarrow_{(\nu=1,2,\ldots)}0$. Hence, $\lim_{\nu\to\infty} q(x_\nu)=0$.

We obtain a generalization of a theorem proved in [4].

THEOREM 9.9. *If S is a sequentially continuous linear lattice and if there exists a monotone, sequentially continuous pure quasi-norm defined on S, then S is super-universally continuous. (A linear lattice S is super-universally continuous if, for every family $x_\lambda \in S$ $(\lambda\in\Lambda)$ satisfying*

(1) $x_\lambda \geqq 0$ for all $\lambda\in\Lambda$,

(2) $\lambda_1,\lambda_2\in\Lambda$ then there is $\lambda_3\in\Lambda$ such that $x_{\lambda_1}\wedge x_{\lambda_2}\geqq x_{\lambda_3}$, there exists a sequence $x_{\lambda_\nu}\in S$ $(\nu=1,2,\cdots)$ such that $\bigwedge_{\lambda\in\Lambda} x_\lambda$ exists and equals $\bigwedge_{\nu=1}^{\infty} x_{\lambda_\nu}$.)

Proof. Let x_λ $(\lambda\in\Lambda)$ be such a family of positive elements. Let

$$A = \inf_{\lambda\in\Lambda}\left\{\sup_{\{\beta;x_\lambda\geqq x_\beta\}} q(x_\lambda - x_\beta)\right\}.$$

To prove $A=0$ we suppose there is $\varepsilon>0$ such that $A>\varepsilon$ and derive a contradiction. If $A>\varepsilon$ there is a sequence $b_1\geqq b_2\geqq\cdots\geqq b_\nu\geqq\cdots$ with $b_\nu\in\{x_\lambda:\lambda\in\Lambda\}$ for $\nu=1,2,\cdots$ such that $q(b_\nu-b_{\nu+1})>\varepsilon$ for $\nu=1,2,\cdots$.

Now let $b_0=\bigwedge_{\nu=1}^{\infty} b_\nu$ which exists since S is sequentially continuous. Since q is monotone $b_\nu-b_{\nu+1}\leqq b_\nu-b_0$ implies $q(b_\nu-b_{\nu+1})\leqq q(b_\nu-b_0)$. Therefore $q(b_\nu-b_0)>\varepsilon$ for $\nu=1,2,\cdots$.

But this contradicts the fact that q is sequentially continuous since $b_\nu-b_0\downarrow_{(\nu=1,2,\ldots)}0$ but $q(b_\nu-b_0)>\varepsilon$ for $\nu=1,2,\cdots$.

Thus, we must have $A=0$. This allows us to select a sequence of elements $x_1',x_2',\cdots$ with $x_\nu'\in\{x_\lambda:\lambda\in\Lambda\}$ for $\nu=1,2,\cdots$ such that

$$\sup_{\{\beta:x_\nu'\geqq x\beta\}} q(x_\nu'-x_\beta) < \frac{1}{2^\nu}\,(\nu=1,2,\cdots).$$

Since $\{x_\lambda:\ \lambda\in\Lambda\}$ is a decreasing system, there exists a sequence $x_1\geqq x_2\geqq\cdots\geqq x_\nu\geqq\cdots$ with $x_\nu\in\{x_\lambda:\ \lambda\in\Lambda\}$ for all $\nu=1,2,\cdots$ such that

$x'_\nu \geqq x_\nu$ $(\nu = 1, 2, \cdots)$. (To get such a sequence we need only let $x_1 = x'_1$ and choose $x_{\nu+1} \leqq x_1 \wedge x_2 \wedge \cdots \wedge x_\nu \wedge x'_{\nu+1}$. Then since q is monotone we have:

$$\text{(1)} \qquad \sup_{\{\beta : x_\nu \geqq x_\beta\}} q(x_\nu - x_\beta) \leqq \sup_{\{\beta : x_\nu \geqq x_\beta\}} q(x'_\nu - x_\beta) < \frac{1}{2^\nu}$$

Now let $x_0 = \bigwedge_{\nu=1}^{\infty} x_\nu$. Since q is sequentially continuous $q(x_\nu - x_0) \leqq (1/2^\nu)$ $(\nu = 1, 2, \cdots)$.

Given any $\lambda \in \Lambda$ we have:

$$\text{(2)} \qquad q((x_\nu \wedge x_\lambda) - (x_0 \wedge x_\lambda)) \leqq q(x_\nu - x_0) \leqq \frac{1}{2^\nu}.$$

Since $\{x_\lambda : \lambda \in \Lambda\}$ is a decreasing system, there is $\beta \in \Lambda$ such that $x_\nu \wedge x_\lambda \geqq x_\beta$. Then, according to (1)

$$q(x_\nu - (x_\nu \wedge x_\lambda)) \leqq q(x_\nu - x_\beta) < \frac{1}{2^\nu}.$$

Hence (2) gives us

$$q(x_\nu - (x_0 \wedge x_\nu)) \leqq q(x_\nu - (x_\nu \wedge x_\lambda)) + q\,((x_\nu \wedge x_\lambda) - (x_0 \wedge x_\lambda)) \leqq \frac{1}{2^{\nu-1}}.$$

Therefore, since q is sequentially continuous $q(x_0 - (x_0 \wedge x_\lambda)) = 0$. But then the fact that q is pure implies that $x_0 = x_0 \wedge x_\lambda$ or $x_0 \leqq x_\lambda$. Since λ was arbitrary we have $x_0 \leqq x_\lambda$ for all $\lambda \in \Lambda$ which gives, us since

$$x_0 = \bigwedge_{\nu=1}^{\infty} x_\nu \ (x_\nu \in \{x_\lambda : \lambda \in \Lambda\} \text{ for all } \nu = 1, 2, \cdots), \ x_0 = \bigwedge_{\lambda \in \Lambda} x_\lambda.$$

10. **Monotone ideals.** An ideal $\boldsymbol{I}$ is said to be *semimonotone* if $B_a \succ \boldsymbol{I}$ for all $a \in S$. The following result follows easily from this definition.

THEOREM 10.1. *$\boldsymbol{I}$ is semimonotone if and only if every quasi-norm in $\boldsymbol{I}$ is semimonotone.*

An ideal $\boldsymbol{I}$ is said to be *monotone* if $\boldsymbol{I}$ has a basis consisting of monotone quasi-norms.

THEOREM 10.2. *If $\boldsymbol{I}$ is monotone and proper, then it is semimonotone.*

Proof. Let $q \in \boldsymbol{I}$. Then there exists $q_1 \in \boldsymbol{I}$ such that q_1 is monotone and $q \prec q_1$. Then, for $a \in S$, B_a is bounded by q_1 since $q_1(B_a) = q_1(a)$ and q_1 is proper. $B_a \succ q_1 \succ q$ shows that $B_a \succ q$.

For any manifold $A \subset S$ we let $B_A = \bigcup_{a \in A} B_a = \{x : |x| \leqq |a| \text{ for some } a \in A\}$.

THEOREM 10.3. *An ideal I on a linear lattice S is monotone if and only if for each $q_1 \in I$ and $\delta_1 > 0$ there is $q_2 \in I$ and $\delta_2 > 0$ such that $B_{\{x:q_2(x)<\delta_2\}} \subset \{x:q_1(x) < \delta_1\}$. (In the case when I is proper this is equivalent to the requirement that, in the corresponding linear topology, there is a basis $\mathfrak{B}$ of neighborhoods of zero such that $V \in \mathfrak{B}$ implies $B_V = V$.)*

Proof. If I is monotone, consider any $q_1 \in I$. Then, by assumption, there is a monotone $q_2 \in I$ such that $q_1 \prec q_2$. For any $\delta_1 > 0$ there is a $\delta_2 > 0$ such that $q_2(x) < \delta_2$ implies $q_1(x) < \delta_1$. Let $A = \{x:q_2(x) < \delta_2\}$. Then $B_A = A$ since, if $x \in A$ and $|y| \leqq |x|$, then $q_2(y) \leqq q_2(x) \leqq \delta_2$. Hence

$$B_{\{x:q_2(x)<\delta_2\}} = \{x:q_2(x) < \delta_2\} < \{x:q_1(x) < \delta_1\}.$$

For the converse we assume the condition is satisfied and show that I is monotone. For $q \in I$ let $\bar{q}$ be defined as before by $\bar{q}(x) = q(B_x)$. In Theorem 8.2 it was shown that $\bar{q}$ is a monotone quasi-norm. It is clear that $\bar{q}(x) \geqq q(x)$ for all $x \in S$. If we can show that $\bar{q} \in I$ for all $q \in I$, then $\{\bar{q}:q \in I\}$ will be a basis of monotone quasi-norms.

For $q \in I$ and $\varepsilon_n = 1/n$ there is a $q_n \in I$, and $\delta_n > 0$ such that $q_n(x) < \delta_n$ and $|y| \leqq |x|$ imply $q(y) < 1/n$.

Therefore $q_n(x) < \delta_n$ implies $\bar{q}(x) = q(B_x) \leqq 1/n$. Since I is an ideal, there is $p \in I$ such that $p \succ q_n$ for all $n = 1,2,\cdots$. Then, for any $\varepsilon > 0$, there is a positive integer n_0 such that $1/n_0 < \varepsilon$. We know $q_{n_0}(x) < \delta_{n_0}$ implies $\bar{q}(x) \leqq 1/n_0$. For this δ_{n_0} there is $\delta > 0$ such that $p(x) < \delta$ implies $q_{n_0}(x) < \delta_{n_0}$. Therefore $p(x) < \delta$ implies $\bar{q}(x) < \varepsilon$. Hence $\bar{q} \prec p \in I$ and this implies $\bar{q} \in I$.

If q_1 and q_2 are quasi-norms such that $q_1 \prec q_2$, then $\bar{q}_1 \prec \bar{q}_2$. This follows by Theorem 8.3 since $\bar{q}_2 \succ q_2 \succ q_1$ and $\bar{q}_2$ is monotone. Given an ideal I consider $\{\bar{q}:q \in I\}$. This system of quasi-norms satisfies the basis condition since, if $\bar{q}_\nu$ $(\nu = 1,2,\cdots)$ is a sequence from $\{\bar{q}:q \in I\}$, then there is a $q \in I$ such that $q \succ q_\nu$ $(\nu = 1,2,\cdots)$ which implies $\bar{q} \succ \bar{q}_\nu$ $(\nu = 1,2,\cdots)$. The ideal $\bar{I}$ which has $\{\bar{q}:q \in I\}$ for a basis is called the *monotone hull of I*.

THEOREM 10.4. *If I is an ideal contained in a monotone ideal I_1, then $\bar{I} \subset I_1$. (It follows from this that $\bar{I} = \bigwedge_{I \subset I_1 \text{ monotone}} I_1$.)*

Proof. If $q \in I$, then there exists a monotone $q_1 \in I$ such that $q \prec q_1$. Thus, we have $\bar{q} \prec q_1$ which implies $\bar{q} \in I_1$. Therefore $\bar{I} \subset I_1$.

For any $a \in S$, B_a is of character 1. Obviously B_a is star and symmetric and, for $x, y \in B_a$, $\lambda + \mu \leqq 1$, $\lambda, \mu \geqq 0$, we have

$$|\lambda x + \mu y| \leqq |\lambda x| + |\mu y| \leqq |\lambda + \mu|\,|a| \leqq |a|.$$

If I is any semimonotone ideal, we know that $B_a \succ I$ for all $a \in S$. In other words, $B_a \succ q$ for all $a \in S$ and $q \in I$. But if q_{B_a} is the quasi-norm associated with the finite character set B_a, Theorem 2.10 shows that $q_B \succ q$ for all $q \in I$

and all $a \in S$. In other words, $q_{B_a} \in \boldsymbol{I}^{\mathfrak{F}}$ for all $a \in S$. Now let $F = \{q : q \succ q_{B_a}$ for some $a \in S\}$. Then we see that $\boldsymbol{F} \subset \boldsymbol{I}^{\mathfrak{F}}$ implies $\boldsymbol{F}^{\mathfrak{Z}} \supset \boldsymbol{I}^{\mathfrak{F}\mathfrak{Z}} \supset \boldsymbol{I}$ for any semimonotone ideal $\boldsymbol{I}$. If $q \in \boldsymbol{F}^{\mathfrak{Z}}$, then $q \prec q_{B_a}$ for all $a \in S$ which implies $B_a \succ q$ for all $a \in S$ and this means q is semimonotone. Therefore $\boldsymbol{F}^{\mathfrak{Z}}$ is semimonotone itself and it is clearly the strongest semimonotone ideal. Since B_a is of finite character for every $a \in S$ we see that $\boldsymbol{I}_1 \sim \boldsymbol{I}_2$ and $\boldsymbol{I}_1$ semimonotone implies $\boldsymbol{I}_2$ is semimonotone. Then we conclude that $\boldsymbol{I} \sim \boldsymbol{F}^{\mathfrak{Z}}$ implies that $\boldsymbol{I}$ is semimonotone which means $\boldsymbol{I} \subset \boldsymbol{F}^{\mathfrak{Z}}$. Thus, $\boldsymbol{F}^{\mathfrak{Z}}$ is equivalently strongest (i.e. $\boldsymbol{F}^{\mathfrak{Z}} = \bigvee_{\boldsymbol{I} \sim \boldsymbol{F}^{\mathfrak{Z}}} \boldsymbol{I}$). Also $\boldsymbol{F}^{\mathfrak{Z}}$ is clearly reflexive.

THEOREM 10.5. *If **I** is a semimonotone ideal, then the equivalent hull of **I** and the reflexive extension of **I** are also semimonotone.*

Proof. If $\boldsymbol{I}_1 \sim \boldsymbol{I}$, then $\boldsymbol{I}_1$ is semimonotone which implies $\boldsymbol{I}_1 \subset \boldsymbol{F}^{\mathfrak{Z}}$. Therefore $\bigvee_{\boldsymbol{I}_1 \sim \boldsymbol{I}} \boldsymbol{I}_1 \subset \boldsymbol{F}^{\mathfrak{Z}}$ which shows that the equivalent hull is semimonotone. Similarly $\boldsymbol{I} \subset \boldsymbol{F}^{\mathfrak{Z}}$ gives $\boldsymbol{I}^{\mathfrak{F}\mathfrak{Z}} \subset \boldsymbol{F}^{\mathfrak{Z}\mathfrak{F}\mathfrak{Z}} = \boldsymbol{F}^{\mathfrak{Z}}$ and this shows that $\boldsymbol{I}^{\mathfrak{F}\mathfrak{Z}}$ is semimonotone.

An ideal $\boldsymbol{I}$ is said to be *sequentially continuous* if $\boldsymbol{I}$ contains a basis consisting of sequentially continuous quasi-norms. Theorem 9.5 then shows that in a sequentially continuous ideal every quasi-norm is sequentially continuous.

THEOREM 10.6. *If **I** is sequentially continuous and S is Archimedean, then **I** is semimonotone.*

Proof. This follows easily from Theorem 9.6.

We say that an ideal is semicontinuous if it contains a basis consisting of semicontinuous quasi-norms.

Using the terminology in [5] we have:

THEOREM 10.7. *If q is a semicontinuous quasi-norm on a universally continuous linear lattice S, there is a unique normal manifold $N \subset S$ such that q is pure in N and $q(x) = 0$ for all $x \in N^{\perp}$.*

Proof. Let $M = \{x : q(x) = 0\}$. Then it is easily seen that M is a linear manifold. Since q is monotone, $|x| \leqq |y|$ and $y \in M$ implies $x \in M$. Therefore M is a seminormal manifold. If $x_\lambda \uparrow_{\lambda \in \Lambda} x$ and $x_\lambda \in M$ for all $\lambda \in \Lambda$, then $q(x) = \sup_{\lambda \in \Lambda} q(x_\lambda) = 0$ by assumption and $x \in M$. Hence, by a theorem in linear lattices M is normal. Clearly $q(x) = 0$ for all $x \in M$. Also $q(x) = 0$ implies that $x \in M$ which means $[M^{\perp}]x = 0$. Therefore q is pure in $M^{\perp}$. To show that M is unique, we suppose q is pure in a normal manifold P and $q(x) = 0$ for all $x \in P^{\perp}$. Then clearly $P^{\perp} = M$.

THEOREM 10.8. *Given any system of semicontinuous quasi-norms q_λ $(\lambda \in \Lambda)$ on a universally continuous linear lattice S there is a unique normal manifold $N \subset S$ such that q_λ $(\lambda \in \Lambda)$ satisfies $q_\lambda(x) = 0$ for all $\lambda \in \Lambda$ if and only if $x \in N$.*

Proof. Let $N_\lambda = \{x : q_\lambda(x) = 0\}$ and let $N = \bigcap_{\lambda \in \Lambda} N_\lambda$. Then N is the intersection of normal manifolds and is therefore normal itself. If $x \neq 0$ and $x \in N^\perp$, then by definition of N there is $\lambda \in \Lambda$ such that $q_\lambda(x) \neq 0$.

THEOREM 10.9. *Let S be a sequentially continuous linear lattice. Suppose there is a pure, proper, semicontinuous quasi-norm q defined on S. Then, for all $a \in S$, B_a is complete by the uniformity induced by $\{q\}^3$.*

Proof. Let $B_a \ni x_\delta$ $(\delta \in \Delta)$ be a Cauchy system. Then, by Theorem 5.3, for every positive integer ν there is $\delta'_\nu \in \Delta$ such that $q(x_{\delta_1} - x_{\delta_2}) \leqq 1/2^\nu$ for all $\delta_1, \delta_2 \leqq \delta'_\nu$. Since Δ is a directed system we can define by induction a sequence $\delta_\nu \in \Delta$ $(\nu = 1, 2, \cdots)$ such that $\delta_\nu \leqq \delta_\nu, \delta'_{\nu+1}$. For ease of notation, we write $a_\nu = x_{\delta_\nu}$ $(\nu = 1, 2, \cdots)$. Now consider

$$q\left(\sum_{\nu=\mu}^{\sigma} |a_{\nu+1} - a_\nu|\right) \leqq \sum_{\nu=\mu}^{\sigma} q(a_{\nu+1} - a_\nu) \leqq \sum_{\nu=\mu}^{\sigma} \frac{1}{2^\nu},$$

$$0 \leqq \bigvee_{\nu=\mu}^{\sigma} a_\nu - a_\mu = \bigvee_{\nu=\mu}^{\sigma} (a_\nu - a_\mu) \leqq \sum_{\nu=\mu}^{\sigma} |a_{\nu+1} - a_\nu|.$$

This last follows since, for $\mu \leqq \nu \leqq \sigma$, we have

$$a_\nu - a_\mu \leqq |a_\nu - a_\mu| = |a_\nu - a_{\nu-1} + a_{\nu-1} - a_{\nu-2} + \cdots + a_{\mu+1} - a_\mu|$$
$$\leqq \sum_{\nu=\mu}^{\sigma} |a_{\nu+1} - a_\nu|.$$

Then, since q is monotone,

$$q\left(\bigvee_{\nu=\mu}^{\sigma} a_\nu - a_\mu\right) \leqq q\left(\sum_{\nu=\mu}^{\sigma} |a_{\nu+1} - a_\nu|\right) \leqq \sum_{\nu=\mu}^{\sigma} \frac{1}{2^\nu} \leqq \frac{1}{2^{\mu-1}}.$$

Therefore, since S is sequentially continuous, $|x_\delta| \leqq |a|$ $(\delta \in \Delta)$ and q is semicontinuous;

$$q\left(\bigvee_{\nu=\mu}^{\infty} a_\nu - a_\mu\right) = \sup_{\sigma=\mu,\mu+1,\dots} q\left(\bigvee_{\nu=\mu}^{\sigma} a_\nu - a_\mu\right) \leqq \frac{1}{2^{\mu-1}}.$$

Similarly we find $q(a_\mu - \bigwedge_{\nu=\mu}^{\infty} a_\nu) \leqq 1/2^{\mu-1}$.

Thus $q(\bigvee_{\nu=\mu}^{\infty} a_\nu - \bigwedge_{\nu=\mu}^{\infty} a) \leqq 1/2^{\mu-2}$.

Now let $l_\mu = \bigvee_{\nu=\mu}^{\infty} a_\nu - \bigwedge_{\nu=\mu}^{\infty} a_\nu$ and set $l = \bigwedge_{\mu=1}^{\infty} l_\mu$.

Then $0 \leqq l \leqq l_\mu$ implies $q(l) \leqq q(l_\mu) \leqq 1/2^{\mu-2}$. Since this is true for all $\mu = 1, 2, \cdots$ we see that $q(l) = 0$ which means, since q is pure, that $l = 0$. Therefore there is $a_0 \in S$ such that order-$\lim_{\nu\to\infty} a_\nu = a_0$.

Given any $\varepsilon > 0$ since $\bigvee_{\nu=\mu}^{\infty} a_\nu \geqq a_0 \geqq \bigwedge_{\nu=\mu}^{\infty} a_\nu$ we have:

$$|a_0 - a_\mu| \leqq \left(\bigvee_{\nu=\mu}^{\infty} a_\nu - a_\mu\right) \vee \left(a_\mu - \bigwedge_{\nu=\mu}^{\infty} a_\nu\right) \leqq l_\mu$$

which implies $q(a_0 - a_\mu) \leqq 1/2^{\mu-2}$. Now choose μ_0 such that $5/2^{\mu_0} < \varepsilon$. Then, for $\delta \leqq \delta_{\mu_0}$, we have

$$q(a_0 - x_\delta) \leqq q(a_0 - x_{\delta_{\mu_0}}) + q(x_{\delta_{\mu_0}} - x_\delta)$$
$$= q(a_0 - a_{\mu_0}) + q(x_{\delta_{\mu_0}} - x_\delta) \leqq 1/2^{\mu_0 - 2} + 1/2^{\mu_0} = 5/2^{\mu_0} < \varepsilon.$$

Therefore $x_\delta \to_{\delta \in \Delta} a_0$.

As a generalization of Theorem 3.4 in [6], we have:

THEOREM 10.10. *Let S be a universally continuous linear lattice. Let $\boldsymbol{I}$ be a proper semicontinuous ideal defined on S. Then, for any $a \in S$, B_a is complete by $\mathfrak{U}^{\mathfrak{Z}}$.*

Proof. Suppose x_δ $(\delta \in \Delta)$ is a Cauchy system in B_a. Let $\boldsymbol{L}$ be a basis for $\boldsymbol{I}$ composed of semicontinuous quasi-norms and choose $q \in \boldsymbol{L}$. Then, by Theorem 9.6, there is a unique normal manifold N_q of S such that q^{N_q} is pure in N_q and $q(x) = 0$ for all $x \in N_q^\perp$. Then q^{N_q} is a semicontinuous, pure quasi-norm on N_q.

$$q([N_q](x_{\delta_1} - x_{\delta_2})) = q(|[N_q](x_{\delta_1} - x_{\delta_2})|) \leqq q(|x_{\delta_1} - x_{\delta_2}|)$$
$$= q(x_{\delta_1} - x_{\delta_2}) \to_{\delta_1, \delta_2 \in \Delta} 0.$$

This shows that $[N_q]x_\delta$ $(\delta \in \Delta)$ is a Cauchy system in N_q by the uniformity induced by q^{N_q}. Thus, by the previous theorem there is an $x_q \in N_q$ such that $[N_q]x_\delta \to_{\delta \in \Delta} x_q$.

Now let $q_1, q_2 \in \boldsymbol{L}$.

$$\begin{aligned}[N_{q_1}][N_{q_2}]x_{q_1} &= [N_{q_1}][N_{q_2}] \lim_{\delta \in \Delta} [N_{q_1}]x_\delta \\ &= [N_{q_1}] \lim_{\delta \in \Delta} [N_{q_2}][N_{q_1}]x_\delta \\ &= [N_{q_1}][N_{q_1}] \lim_{\delta \in \Delta} [N_{q_2}]x_\delta \\ &= [N_{q_1}]x_{q_2} = [N_{q_1}][N_{q_2}]x_{q_2}.\end{aligned}$$

Since S is universally continuous and since $|x_q| \leqq a$ for all $q \in \boldsymbol{L}$ we see that $x_1 = \bigvee_{q \in \boldsymbol{L}} x_q^+$ and $x_2 = \bigvee_{q \in \boldsymbol{L}} x_q^-$ both exist.

$$\begin{aligned}[N_p]x_1 &= [N_p] \bigvee_{q \in \boldsymbol{L}} x_q^+ = [N_p] \bigvee_{q \in \boldsymbol{L}} [N_q]x_q^+ = \bigvee_{q \in \boldsymbol{L}} [N_p][N_q]x_q^+ \\ &= \bigvee_{q \in \boldsymbol{L}} [N_p][N_q]x_p^+ = [N_p]x_p^+ = x_p^+.\end{aligned}$$

Similarly $[N_p]x_2 = x_p^-$ and therefore, if $x_0 = x_1 - x_2$ we have $[N_p]x_0 = x_p^+ - x_p^- = x_p$.

Then $x_\delta \to_{\delta\in\Delta} x_0$ since, for any $\varepsilon > 0$, and for any $q \in \boldsymbol{L}$ there is $\delta_0 \in \Delta$ such that

$$q([N_q]x_\delta - x_q) = q([N_q](x_\delta - x_0)) < \varepsilon \text{ for } \delta \leqq \delta_0.$$

But $x_\delta - x_0 = [N_q](x - x_0) + [N_q^\perp]\ (x_\delta - x_0)$ and therefore

$$q(x_\delta - x_0) \leqq q([N_q](x_\delta - x_0)) + q([N^\perp_q](x_\delta - x_0)) \leqq \varepsilon + 0 = \varepsilon$$

for all $\delta \leqq \delta_0$.

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